MODEL ANSWERS TO HWK #9

1. There are a number of ways to proceed; probably the most straightforward is to view the region D as something of type 2:

$$
\begin{aligned}
\iint_D x + y \,\mathrm{d}x\,\mathrm{d}y &= \int_{-1}^{2} \left(\int_{y^2-2y}^{2-y} x + y \,\mathrm{d}x \right) \mathrm{d}y \\
&= \int_{-1}^{2} \left[\frac{x^2}{2} + yx \right]_{y^2-2y}^{2-y} \mathrm{d}y \\
&= \int_{-1}^{2} \frac{(2-y)^2}{2} + y(2-y) - \frac{(y^2-2y)^2}{2} - y(y^2-2y)\,\mathrm{d}y \\
&= \int_{-1}^{2} -\frac{y^4}{2} + y^3 - \frac{y^2}{2} + 2\,\mathrm{d}y \\
&= \left[-\frac{y^5}{2\cdot 5} + \frac{y^4}{4} - \frac{y^3}{2\cdot 3} + 2y \right]_{-1}^{2} \\
&= -\frac{2^4}{5} + 2^2 - \frac{2^2}{3} + 2^2 - \frac{1}{10} - \frac{1}{4} - \frac{1}{6} + 2 \\
&= \frac{99}{20}.
\end{aligned}
$$

2. There are a number of ways to proceed; probably the most straightforward is to view the region D as something of type 1:

$$
\begin{aligned}
\iint_D 3y \,\mathrm{d}x\,\mathrm{d}y &= \int_0^{\frac{1}{9}} \left(\int_x^3 3y \,\mathrm{d}y \right) \mathrm{d}x + \int_{\frac{1}{9}}^{1} \left(\int_x^{x^{-1/2}} 3y \,\mathrm{d}y \right) \mathrm{d}x \\
&= \int_0^{\frac{1}{9}} \left[\frac{3y^2}{2} \right]_x^3 \mathrm{d}x + \int_{\frac{1}{9}}^{1} \left[\frac{3y^2}{2} \right]_x^{x^{-1/2}} \mathrm{d}x \\
&= \int_0^{\frac{1}{9}} \frac{3^3}{2} - \frac{3x^2}{2}\,\mathrm{d}x + \int_{\frac{1}{9}}^{1} \frac{3}{2x} - \frac{3x^2}{2}\,\mathrm{d}x \\
&= \left[\frac{3^3 x}{2} - \frac{x^3}{2} \right]_0^{\frac{1}{9}} + \left[\frac{3}{2}\ln x - \frac{x^3}{2} \right]_{\frac{1}{9}}^{1} \\
&= \frac{3}{2} - \frac{1}{2\cdot 3^6} - \frac{1}{2} + 3\ln 3 + \frac{1}{2\cdot 3^6} \\
&= 1 + 3\ln 3.
\end{aligned}
$$

3.

$$\begin{aligned}
\int_0^2 \left(\int_0^{4-y^2} x \, \mathrm{d}x \right) \mathrm{d}y &= \int_0^2 \left[\frac{x^2}{2} \right]_0^{4-y^2} \mathrm{d}y \\
&= \int_0^2 \frac{(4-y^2)^2}{2} \, \mathrm{d}y \\
&= \int_0^2 8 - 4y^2 + \frac{y^4}{2} \, \mathrm{d}y \\
&= \left[8y - \frac{4y^3}{3} + \frac{y^5}{2 \cdot 5} \right]_0^2 \\
&= 16 - \frac{32}{3} + \frac{2^4}{5} \\
&= \frac{2^4 \cdot 3 \cdot 5 - 2^5 \cdot 5 + 2^4 \cdot 3}{3 \cdot 5} \\
&= \frac{2^4 \cdot 3 \cdot 6 - 2^5 \cdot 5}{3 \cdot 5} \\
&= \frac{2^5 (9 - 5)}{3 \cdot 5} \\
&= \frac{2^7}{3 \cdot 5}.
\end{aligned}$$

The region in question is bounded by the curves $x = 0$, $y = 0$ and $y^2 = 4 - x$. So, reversing the order of integration, we get

$$\begin{aligned}
\int_0^4 \left(\int_0^{\sqrt{4-x}} x \, \mathrm{d}y \right) \mathrm{d}x &= \int_0^4 x \, [y]_0^{\sqrt{4-x}} \, \mathrm{d}x \\
&= \int_0^4 x \sqrt{4-x} \, \mathrm{d}x \\
&= \left[-\frac{2x}{3} (4-x)^{3/2} \right]_0^4 + \int_0^4 \frac{2}{3} (4-x)^{3/2} \, \mathrm{d}x \\
&= \left[-\frac{4}{3 \cdot 5} (4-x)^{5/2} \right]_0^4 \\
&= \frac{2^7}{3 \cdot 5}.
\end{aligned}$$

4.

$$\int_0^8 \left(\int_0^{\sqrt{y/3}} y \,\mathrm{d}x\right) \mathrm{d}y + \int_8^{12} \left(\int_{\sqrt{y-8}}^{\sqrt{y/3}} y \,\mathrm{d}x\right) \mathrm{d}y = \int_0^2 \left(\int_{3x^2}^{x^2+8} y \,\mathrm{d}y\right) \mathrm{d}x$$
$$= \int_0^2 \left[\frac{y^2}{2}\right]_{3x^2}^{x^2+8} \mathrm{d}x$$
$$= \int_0^2 \frac{(x^2+8)^2}{2} - \frac{(3x^2)^2}{2} \,\mathrm{d}x$$
$$= \left[\frac{x^5}{2\cdot 5} + \frac{8x^3}{3} + 2^5 x - \frac{9x^5}{2\cdot 5}\right]_0^2$$
$$= \frac{2^4}{5} + \frac{2^6}{3} + 2^6 - \frac{9\cdot 2^4}{5}$$
$$= \frac{896}{15}.$$

5. This is a region of type 4; we view this as an elementary region of type 1. The projection of W onto the xy-plane is the elementary region of type 2 bounded by $y = x^2$ and $y = 9$.

$$\iiint_W 8xyz \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = \int_{-3}^3 \left(\int_{x^2}^9 \left(\int_0^{9-y} 8xyz \,\mathrm{d}z\right) \mathrm{d}y\right) \mathrm{d}x$$
$$= 8\int_{-3}^3 x \left(\int_{x^2}^9 y \left(\int_0^{9-y} z \,\mathrm{d}z\right) \mathrm{d}y\right) \mathrm{d}x$$
$$= 8\int_{-3}^3 x \left(\int_{x^2}^9 y \left[\frac{z^2}{2}\right]_0^{9-y} \mathrm{d}y\right) \mathrm{d}x$$
$$= 8\int_{-3}^3 x \left(\int_{x^2}^9 \frac{y(9-y)^2}{2} \,\mathrm{d}y\right) \mathrm{d}x$$
$$= 4\int_{-3}^3 x \left(\int_{x^2}^9 81y - 18y^2 + y^3 \,\mathrm{d}y\right) \mathrm{d}x$$
$$= 4\int_{-3}^3 x \left[\frac{81y^2}{2} - 6y^3 + \frac{y^4}{4}\right]_{x^2}^9 \mathrm{d}x$$
$$= 4\int_{-3}^3 \left(\frac{3^8}{2} - 2\cdot 3^7 + \frac{3^8}{4}\right) x - \frac{81x^3}{2} + 6x^7 - \frac{x^9}{4} \mathrm{d}x$$
$$= 0,$$

as x, x^3, x^7 and x^9 are all odd functions. In retrospect, we could have decide very early on that the integral is zero;

$$J(x) = \int_{x^2}^{9} y \left(\int_0^{9-y} z \,\mathrm{d}z \right) \mathrm{d}y,$$

is clearly an even function of x, so that $xJ(x)$ is an odd function.

6. This is a region of type 4; we view this as an elementary region of type 1. The projection of W onto the xy-plane is the elementary region of type 2 bounded by $x = 0$, $y = 3$ and $y = x$.

$$\begin{aligned}
\iiint_W z \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^3 \left(\int_x^3 \left(\int_0^{\sqrt{9-y^2}} z \,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x \\
&= \int_0^3 \left(\int_x^3 \left[\frac{z^2}{2} \right]_0^{\sqrt{9-y^2}} \mathrm{d}y \right) \mathrm{d}x \\
&= \int_0^3 \left(\int_x^3 \frac{9-y^2}{2} \,\mathrm{d}y \right) \mathrm{d}x \\
&= \frac{1}{2} \int_0^3 \left[9y - \frac{y^3}{3} \right]_x^3 \mathrm{d}x \\
&= \frac{1}{2} \int_0^3 18 - 9x + \frac{x^3}{3} \,\mathrm{d}x \\
&= \frac{1}{2} \left[18x - \frac{9x^2}{2} + \frac{x^4}{12} \right]_0^3 \\
&= 3^3 - \frac{3^4}{4} + \frac{3^3}{8} \\
&= \frac{3^3}{8}(8 - 6 + 1) \\
&= \frac{81}{8}.
\end{aligned}$$

7. This is the region bounded by the planes $y = \pm 1$, $x = y^2$, $z = 0$ and $x + z = 1$. So the other five ways to write this region are:

$$\int_0^1 \left(\int_{-\sqrt{x}}^{\sqrt{x}} \left(\int_0^{1-x} f(x, y, z) \, \mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x$$
$$\int_0^1 \left(\int_0^{1-x} \left(\int_{-\sqrt{x}}^{\sqrt{x}} f(x, y, z) \, \mathrm{d}y \right) \mathrm{d}z \right) \mathrm{d}x$$
$$\int_0^1 \left(\int_0^{1-z} \left(\int_{-\sqrt{x}}^{\sqrt{x}} f(x, y, z) \, \mathrm{d}y \right) \mathrm{d}x \right) \mathrm{d}z$$
$$\int_{-1}^1 \left(\int_0^{1-y^2} \left(\int_{y^2}^{1-z} f(x, y, z) \, \mathrm{d}x \right) \mathrm{d}z \right) \mathrm{d}y$$
$$\int_0^1 \left(\int_{\sqrt{1-z}}^{\sqrt{1-z}} \left(\int_{y^2}^{1-z} f(x, y, z) \, \mathrm{d}x \right) \mathrm{d}y \right) \mathrm{d}z.$$

8. T is a linear transformation; therefore it takes straight lines to straight lines. So D is the parallelogram with vertices

$$T(0, 0) = (0, 0) \quad T(1, 3) = (11, 2) \quad T(-1, 2) = (4, 3) \quad T(0, 5) = (15, 5).$$

9. Since T is supposed to take $(0, 5)$ to $(4, 1)$, it must take $(0, 1)$ to $(4/5, 1/5)$. Since T is supposed to take $(-1, 3)$ to $(3, 2)$ and $(1, 2)$ to $(1, -1)$ it should take

$$(5, 0) = 3(1, 2) - 2(-1, 3),$$

to

$$3(3, 2) - 2(1, -1) = (7, 8).$$

Therefore

$$T(1, 0) = (7/5, 8/5).$$

Therefore

$$T(u, v) = \begin{pmatrix} 7/5 & 4/5 \\ 8/5 & 1/5 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

10. We have $x = u$ and $y = (v + u)/2$. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)}(u, v) = \begin{vmatrix} 1 & 0 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

This is nowhere zero. As the map is linear, it follows that the map is injective, and so by the Inverse function theorem it defines a diffeomorphism. Therefore

$$\begin{aligned}\int_0^2 \left(\int_{x/2}^{(x/2)+1} x^5 (2y - x) e^{(2y-x)^2} \, \mathrm{d}x \right) \mathrm{d}y &= \frac{1}{2} \int_0^2 \left(\int_0^2 u^5 v e^{v^2} \, \mathrm{d}v \right) \mathrm{d}u \\ &= \frac{1}{4} \int_0^2 u^5 \left[e^{v^2} \right]_0^2 \mathrm{d}u \\ &= \frac{e^4 - 1}{4} \int_0^2 u^5 \mathrm{d}u \\ &= \frac{e^4 - 1}{24} \left[u^6 \right]_0^2 \\ &= \frac{8(e^4 - 1)}{3}.\end{aligned}$$

11. Let $u = 2x + y$ and $v = x - y$. Then

$$\frac{\partial(u, x)}{\partial(x, y)}(x, y) = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3.$$

So

$$\frac{\partial(x, y)}{\partial(u, v)}(u, v) = -\frac{1}{3}.$$

This is nowhere zero. As the map is linear, it follows that the map is injective, and so by the Inverse function theorem it defines a diffeomorphism. Therefore,

$$\begin{aligned}\iint_D (2x + y)^2 e^{x-y} \, \mathrm{d}x \, \mathrm{d}y &= \frac{1}{3} \int_1^4 \left(\int_{-1}^1 u^2 e^v \, \mathrm{d}v \right) \mathrm{d}u \\ &= \frac{1}{3} \int_1^4 u^2 \left[e^v \right]_{-1}^1 \mathrm{d}u \\ &= \frac{e - e^{-1}}{3} \int_1^4 u^2 \, \mathrm{d}u \\ &= \frac{e - e^{-1}}{9} \left[u^3 \right]_1^4 \\ &= 7(e - e^{-1}).\end{aligned}$$

12. Let $u = y + 2x$ and $v = 2y - x$. Then D^* is the region

$$[0, 5] \times [-5, 0],$$

in uv-coordinates.

$$\frac{\partial(u, x)}{\partial(x, y)}(x, y) = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5.$$

So
$$\frac{\partial(x,y)}{\partial(u,v)}(u,v) = \frac{1}{5}.$$
This is nowhere zero. As the map is linear, it follows that the map is injective, and so by the Inverse function theorem it defines a diffeomorphism. Therefore
$$\begin{aligned}
\iint_D \frac{2x+y-3}{2y-x+6}\,\mathrm{d}x\,\mathrm{d}y &= \frac{1}{5}\int_0^5 \left(\int_{-5}^0 \frac{u-3}{v+6}\,\mathrm{d}v\right)\mathrm{d}u \\
&= \frac{1}{5}\int_0^5 (u-3)\left[\ln(v+6)\right]_{-5}^0 \mathrm{d}u \\
&= \frac{\ln 6}{5}\int_0^5 (u-3)\,\mathrm{d}u \\
&= \frac{\ln 6}{5}\left[\frac{u^2}{2} - 3u\right]_0^5 \\
&= \ln 6\left(\frac{5}{2} - 3\right) \\
&= -\frac{\ln 6}{2}.
\end{aligned}$$