FINAL EXAM PRACTICE MATERIALS

1. Announcements

- Office Hours the week of May 9th:
- Vera - Wednesday, 3-4 pm; Friday, 2-4 pm
- Christine - Tuesday, 4:30-5:30 pm; Wednesday 9-10 am; Sunday, 5-8 pm
- **Do not** expect posted solutions to all of the problems listed below. There are just too many of them. Talk to one another about your solutions and come to office hours to work them out if you have concerns.
- Exam on May 16, 1:30-4:30 pm, in 2-131
- I will email the exam format to the class by Wednesday of next week.

2. Definitions and Theorems

Note, this list only includes new definitions and theorems (since the previous exam). You should also know all of the ones prior to the new material as well.

(1) State the necessary and sufficient condition for a vector field to be a gradient vector field on an open, simply connected set S.
(2) State Green's Theorem for simply connected and multiply connected domains. (Make sure to state what conditions must hold on the domains in order to apply the theorem.)
(3) State the Change of Variables Theorem.
(4) State Stokes' Theorem for simple smooth surfaces.
(5) State the Divergence Theorem.
(6) Write the formula to calculate surface area of a parametrized surface.
(7) Write the formula to determine the surface integral of a scalar field f on a surface S parametrized from $R \subset \mathbb{R}^2$ by the map $\mathbf{r}$.

3. True or False Problems

Give proofs for your claims. If it follows from a theorem or fact mentioned in the lectures then just state the theorem or fact. If your answer is false then give a counterexample or say why it is nonsense.

Also note that $F \in \mathcal{C}^1$ implies all components have continuous first partials and $F \in \mathcal{C}^2$ implies all components have continuous second partials.

(1) Let f be a scalar field defined on $\mathbb{R}^2$ and suppose directional derivatives of f exist for all directions. Then f is continuous.
(2) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. That is, each component of F is linear. Let M be the matrix representation of F in the standard basis for $\mathbb{R}^3$. Then $DF = M$.
(3) Let F be a $\mathcal{C}^2$ vector field on $\mathbb{R}^3$. Then $\nabla \cdot (\nabla \times F) = 0$.
(4) Let f be a scalar function on $\mathbb{R}^3$ which is in class $\mathcal{C}^2$. Then $\operatorname{curl}(\nabla f) = 0$.
(5) Let F be a $\mathcal{C}^1$ vector field on $\mathbb{R}^3 - \{0\}$. If $\operatorname{curl} F = 0$ then there exists a function f on $\mathbb{R}^3 - \{0\}$ such that $F = \nabla f$. (i.e. F is conservative on this region)

(6) Let S be a smooth surface with a unit normal $\vec{n}$. Let F be a vector field on $\mathbb{R}^3$ where $F = \vec{n}$ on S. Then $\int\int_S F \cdot \vec{n}\, dS = \text{area}\,(S)$.
(7) Let F be a divergence free vector field on an open subset of $\mathbb{R}^3$ (that is, let $divF = 0$). Then there exists a vector field G such that $F = \nabla \times G$.
(8) Let F be a $\mathcal{C}^1$ vector field in $\mathbb{R}^3$ and let f be a scalar function such that ∇f is perpendicular to F. Assume $V = \{f(x, y, z) \leq 1\}$ is a solid where $S = \{f(x, y, z) = 1\}$ is its boundary and S is bounded orientable smooth surface. Then $\int\int\int_V \operatorname{div} F dV = 0$.

4. Practice Problems

(1) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a one-to-one linear transformation. Consider a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of $\mathbb{R}^n$. Prove that $\{T(\mathbf{x}_1), \ldots, T(\mathbf{x}_n)\}$ is contained in a basis for $\mathbb{R}^m$.
(2) Find the maximum and minimum values for the function $f(x, y) = \sin x + \cos y$ on the rectangle $[0, 2\pi] \times [0, 2\pi]$.
(3) Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function with the following properties: g has precisely two zeros at a, b with $g'(a) > 0, g'(b) < 0$.
Define $f(x, y) = \int_x^y g(t)dt$. Find ∇f and $Hess_f$ in terms of g and its derivatives. Prove that f has four critical points and classify them.
(4) Find the volume of the region described by $x^2 + y^2 \leq a^2$ and $x^2 + z^2 \leq a^2$.
(5) Compute the triple integral $\int\int\int_W (1-z^2)\, dxdydz$ where W is the pyramid with top vertex $(0, 0, 1)$ and the base vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$.
(6) Assume the integral
$$\int_0^1 \int_0^x \int_0^y f(x, y, z)\, dzdydx$$
exists and Fubini's Theorem can be used. Write the integral with the integration order $dxdydz$
(7) Use polar coordinates to evaluate $\int\int_R \sqrt{x^2 + y^2}\, dxdy$ where $R = [0, 1] \times [0, 1]$.
(8) Compute $\int\int_R (x+y)^2 e^{x-y} dxdy$ where R is the region bounded by $x+y = 1$, $x + y = 4$, $x - y = -1$ and $x - y = 1$.
(9) Show that the surface described by $z^2 = (x^2 + y^2)^{-1}$, where $1 \leq z < \infty$ can be filled but not painted (i.e. the volume under the graph is finite but the surface area is infinite)!
(10) Let S be the sphere of radius R. Compute the integral $\int\int_S (x^2+y^2+z^2)\, dS$.
(11) Let S be part of the cone $z^2 = x^2 + y^2$ with z between 1 and 2 oriented such that the unit normal goes out of the cone. Compute $\int\int_S F \cdot n\, dS$ where $F(x, y, z) = (x^2, y^2, z^2)$.
(12) Find $\int\int_S (\nabla \times F) \cdot n\, dS$ where S is the ellipsoid $x^2 + y^2 + 5z^2 = 16$ and $F = (\sin(xy), e^{x+y}, xy^3)$. (Here the unit normal goes out of S)
(13) Let $F = (y, -x, zx^3y^2)$. Evaluate $\int\int_S (\nabla \times F) \cdot n\, dS$ where S is the surface $x^2 + y^2 + z^2 = 1$, $z \leq 0$ with the unit normal going out of the sphere.
(14) Let $F = (x^3, y^3, z^3)$. Compute the flux of F going out the unit sphere.
(15) Let S be a closed surface which is the surface boundary of a solid V in $\mathbb{R}^3$. Evaluate $\int\int_S (\vec{r} \cdot n)\, dS$ where the normal is outward.

(16) Evaluate the surface integral $\int\int_S F\cdot n\,dS$ where $F=(1,1,z(x^2+y^2)^2)$ and S is the vertical part of the cylinder $x^2+y^2\le 1$, $0\le z\le 1$ (Not including the caps). Assume the unit normal goes outside the cylinder.

(17) Prove Gauss' Law: Let V be a bounded solid which is bounded by a smooth orientable closed surface $S=\partial V$. Assume $(0,0,0)$ is not on S. Then the flux

$$\int\int_S \frac{\vec{r}\cdot\vec{n}}{r^3}\,dS$$

is zero if $(0,0,0)\notin V$ and it is equal to 4π if $(0,0,0)\in V$.

(18) Using the minimal surface equation, verify that every plane in $\mathbb{R}^3$ is a minimal surface.