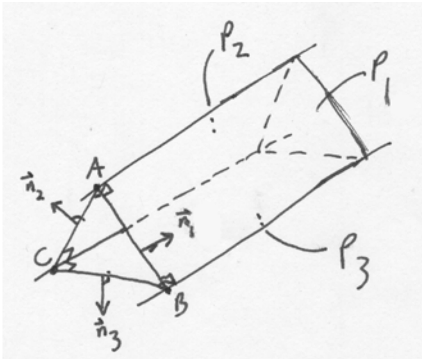
18.02 Problem Set 2, Part II Solutions

1. a)



(b) See the diagram. Because the lines are parallel, we can cut through them with a single plane which is orthogonal to each line. The normal vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ lie in this plane.

(c) One way to do this part is to write

$$(\mathbf{n}_1 \times \mathbf{n}_2) \cdot \mathbf{n}_3 = (c\mathbf{n}_2 \times \mathbf{n}_3) \cdot \mathbf{n}_3 = c((\mathbf{n}_2 \times \mathbf{n}_3) \cdot \mathbf{n}_3) = c \cdot 0 = 0.$$

Here we used that $\mathbf{n}_1 \times \mathbf{n}_2$ is parallel to $\mathbf{n}_2 \times \mathbf{n}_3$ so we can write the latter as a scalar c times the former. Since the triple product of the normals is zero, we know that they are coplanar.

Here is another way:

Each plane P_i can be written as the solutions to an equation $\mathbf{x} \cdot \mathbf{n}_i = a_i$ for scalar a_i.

We are told that the three planes *do not* have a point in common. That means that there is no vector $\mathbf{v}$ solving

$$\begin{pmatrix} \mathbf{n}_1 \\ \hline \mathbf{n}_2 \\ \hline \mathbf{n}_3 \end{pmatrix} \mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

So the determinant of the matrix at left must be zero. But then

$$0 = \begin{vmatrix} \mathbf{n}_1 \\ \hline \mathbf{n}_2 \\ \hline \mathbf{n}_3 \end{vmatrix} = \mathbf{n}_1 \cdot \mathbf{n}_2 \times \mathbf{n}_3.$$

So the three normals are coplanar.

2. (a) In 1 unit of P_1 there are 1/6 of a unit of M_1, 2/6 of a unit of M_2, and 3/6 of a unit of M_3. So to produce p_1 units of P_1, we need $\frac{1}{6}p_1$ units of M_1,

$\frac{1}{3}p_1$ units of M_2, and $\frac{1}{2}p_1$ units of M_3. Thus (using similar reasoning for P_2 and P_3) the linear system is $A\mathbf{p} = \mathbf{m}$ with

$$A = \begin{bmatrix} \frac{1}{6} & \frac{1}{9} & \frac{3}{16} \\ \frac{1}{3} & \frac{1}{3} & \frac{5}{16} \\ \frac{1}{2} & \frac{5}{9} & \frac{1}{2} \end{bmatrix}.$$

(b) Using the method of cofactors we compute $|A| = \frac{1}{864} \neq 0$, and then A^{-1}. We get

$$\mathbf{p} = A^{-1}(\mathbf{m}) = \begin{bmatrix} -6 & 42 & -24 \\ -9 & -9 & 9 \\ 16 & -32 & 16 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}.$$

If $\mathbf{m} = \langle 137, 279, 448 \rangle^T$, then $\mathbf{p} = A^{-1}(\mathbf{m}) = \langle 144, 288, 432 \rangle^{\mathrm{T}}$. So: 144 units of P_1, 288 of P_2, and 432 of P_3 produced.

(c)

$$A = \begin{bmatrix} \frac{1}{6} & \frac{1}{9} & \frac{1}{2}(\frac{1}{6} + \frac{1}{9}) \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{5}{9} & \frac{1}{2}(\frac{1}{2} + \frac{5}{9}) \end{bmatrix}.$$

Here we replaced the third column with the average of the first two. This makes a legitimate set of ratios for composition of P_3 because the entries add up to 1. It also makes the determinant of A equal to zero. Why is this? Well, take the transpose of A and we have a matrix where the third row is the average of the first two rows. So its rows are coplanar. That means it has determinant zero. Taking the transpose preserves the determinant, so the original matrix, shown above, has determinant zero. This means that depending on quantities of raw materials specified, either there will be many solutions (non-uniqueness) or no solutions (inconsistent).

If you didn't think of this method, you could also try putting in variable entries in the third column

$$A = \begin{bmatrix} \frac{1}{6} & \frac{1}{9} & a \\ \frac{1}{3} & \frac{1}{3} & a \\ \frac{1}{2} & \frac{5}{9} & c \end{bmatrix}.$$

and then find various choices for a, b, c which satisfy the equations

$$\begin{aligned} |A| &= 0 \\ a + b + c &= 1. \end{aligned}$$

3. a) Key to this problem: use the cross-product (repeatedly!). There are two planes involved, the plane $\mathcal{P}$ with normal $\mathbf{n}$ and the x-y plane with normal $\mathbf{k}$. Taking the cross product $\mathbf{n} \times \mathbf{k}$ gives a vector which points along the intersection of the two planes. By inspection of the sketch, it is clear that any vector $\mathbf{w}$ which points in the steepest direction of $\mathcal{P}$ lies both in $\mathcal{P}$ and is perpendicular to the line of intersection of $\mathcal{P}$ and the x-y plane. Since $\mathbf{w}$ lies $\mathcal{P}$, it is in perpendicular to $\mathbf{n}$; and since it is perpendicular to the line of intersection of $\mathcal{P}$ and the x-y plane, it is perpendicular to $\mathbf{n} \times \mathbf{k}$. Thus

$$\mathbf{w} = (\mathbf{n} \times \mathbf{k}) \times \mathbf{n}$$

is a vector in $\mathcal{P}$ which points in the steepest direction of $\mathcal{P}$.
b) For the plane containing $\mathbf{u}$ and $\mathbf{v}$ we can take $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ as a normal vector. Substituting this in the answer to part(a), we get

$$\mathbf{w} = ((\mathbf{u} \times \mathbf{v}) \times \mathbf{k}) \times (\mathbf{u} \times \mathbf{v}).$$

Notes: (i) Obviously taking the products in different orders just introduces a scalar factor. (ii) The formula for $\mathbf{w}$ does not give the zero vector (which would be bad!) because the plane $\mathcal{P}$ is assumed to not be the x-y plane. So we are ensured that $\mathbf{n}$ and $\mathbf{k}$ form a positive angle.