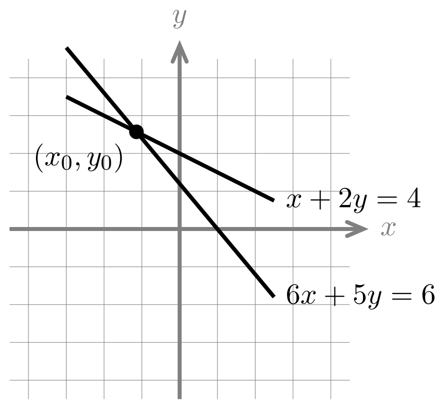
# Geometry of linear systems of equations

Very often in math, science and engineering we need to solve a linear system of equations. A simple example of such a system is given by

$$\begin{array}{rcrcr} 6x & + & 5y & = & 6 \\ x & + & 2y & = & 4. \end{array}$$

You have probably already learned algebraic techniques to solve such a system. Later we will also learn to solve such a system using matrix algebra. For now we will focus on the geometric view of this system.

Solving the system means finding a pair $(x_0, y_0)$ which satisfies both equations. Geometrically each of the equations represents a line. That is, each pair $(x, y)$ satisfying the equation is a point on the line. Thus, the solution $(x_0, y_0)$ is the point where the two lines intersect.

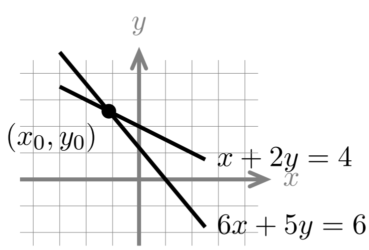



From the graph we can approximate the solution (the exact solution is $(-8/7, 18/7)$), but our interest here is in how many solutions there can be.

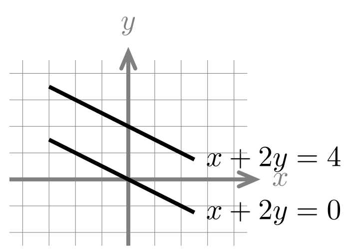
The geometric picture makes this obvious. Here are the three possibilities.

1. The two lines intersect in a point, so there is one solution.
2. The two lines are parallel (and not the same), so there are no solutions.
3. The two lines are the same, so there are an infinite number of solutions.

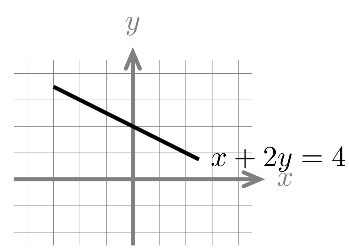
Here are example systems and graphs.




$6x + 5y = 6$
$x + 2y = 4$
(intersecting lines: 1 solution)




$x + 2y = 4$
$x + 2y = 0$
(parallel lines: no solutions)




$x + 2y = 4$
$x + 2y = 4$
(the same line: $\infty$ solutions)

**$3 \times 3$ systems**

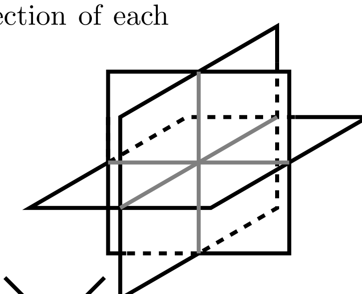
For $3 \times 3$ systems there are more possibilities. For example, consider the system

$$\begin{array}{rcrcrcr} 6x & + & 5y & + & 3z & = & 1 \\ x & + & 2y & + & z & = & 4 \\ 2x & - & 2y & - & 2z & = & 8. \end{array}$$

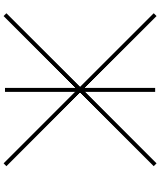
Each equation is the equation of a plane, so, geometrically, solving the system means finding the intersection of three planes, i.e., the point or points which lie on all three planes.

Usually, three planes intersect in a point. You can visualize this by first imagining two of the planes intersecting in a line and then the line intersecting the third plane in a point. Altogether there are four possibilities.

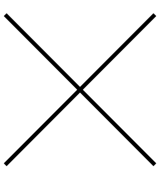
1. Intersect in a point (1 solution to system).
2. Intersect in a line ($\infty$ solutions).
   a) Three different planes, the third plane contains the line of intersection of the first two.
   b) Two planes are the same, the third plane intersects them in a line.
3. Intersect in a plane ($\infty$ solutions)
   a) All three planes are the same.
4. The planes don't all intersect at any point (0 solutions).
   a) The planes are different, but all parallel.
   b) Two planes are parallel, the third crosses them.
   c) The planes are different and none are parallel. but the lines of intersection of each pair are parallel.
   d) Two planes are the same and parallel to the third.



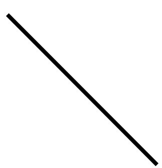
To visualize this we could draw three dimensional figures, for example the figure at the right shows three planes intersecting in a point. Instead, we will visualize the other cases by drawing lines on the page and imagining the planes as extending vertically out of the page.



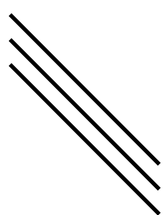
Case (2a)



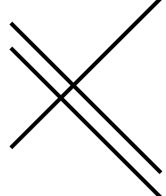
Case (2b)



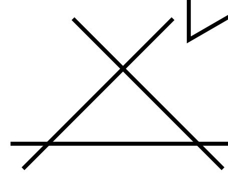
Case (3a)



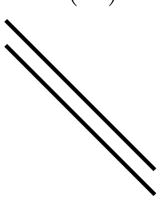
Case (4a)



Case (4b)



Case (4c)



Case (4d)