**JOEL LEWIS:**   Hi. Welcome back to recitation. In lecture, you've been learning about line integrals of vector fields. And I have a couple of nice questions on that subject for you right here.

So I want F to be the vector field whose first coordinate is x*y and whose second coordinate is x squared plus y squared. And so what I'd like you do is compute the line integral of F around two different curves C. So both curves start at the point (1, 1) and they end at the point (2, 4).

So in part a, the curve is just the straight line that connects the point (1, 1) to (2, 4). And in part b, the curve is this sort of piecewise-- it's two sides of a rectangle, right? It goes straight up until it gets to the point (1, 4), and then it goes across to the point (2, 4). So it's a piecewise smooth curve, path, that connects those two points.

So I'd like you to compute the integral over each of these curves of F dot dr. So why don't you pause the video, have a go at that, come back, and we can work it out together.

So, when you're computing a line integral over a curve, really the thing that you want to do is you want to parametrize the curve, and then that gives you stuff that you can plug in. You'll have expressions for x and y in terms of your parameter. So you can plug it in and you just turn this integral right into a nice single variable integral, and then you can compute it. So that's our basic strategy for computing integrals of this form, line integrals of vector fields. So let's have a go. Let's start with part a.

So in part a, what we need to do to apply this method is that we need to parametrize the curve in question. So this is a straight line. And if you look at it, it's the line through the points (1, 1) and (2, 4). So this line has equation y equals 3x minus 2. That's our line, and OK.

So we need to choose some parameter that will give us this segment of this line. So a natural thing to do in this case is-- it's easy enough, y is already written in terms of x, so it's natural enough just to take a parameter that's equal to x. So it's up to you whether you introduce the letter t in order to do this, or not. I'm going to do it with the letter t here in part a, but you could do this problem exactly the same way just using the letter x.

So what I'm going to do is I'm going to let x equals t so that y is equal to 3t minus 2. OK, so that is the parametric equation for the entire line, but we only want the part between the points (1, 1) and (2, 4).

So the part between the lines (1, 1) and (2, 4) is the part where t is between 1 and 2. Where x is between 1 and 2. OK. So this is our parametrization. So now we need to figure out what is the field F in this parametrization, and what is dr. And then after we have those, we can just put them into our integral and compute.

So F, in this parametrization, well, we take the equation for F, which is x*y comma x squared plus y squared, and we just plug in. So in this case, x*y is going to be 3t minus 2 times t, so that's 3 t squared minus 2t. And x squared plus y squared, well that's t squared plus 3t minus 2 quantity squared. So that's what F is.

And also we have that dr-- well, we just take the differentials of x and y-- so this is going to be dt comma 3*dt. Or if you like, 1 comma 3 times dt if you like to factor out your dt. So that's what F and dr is. So now we need to compute our integral.

So the integral over C of F dot dr, well, you just plug in. So this is the integral over C now. So OK. So now we need to look at our bounds.

So the integral over C means the integral as t varies in the range that we need to cover. That whole curve. So in this case, we said that was from t equals 1 to 2. So it's the integral as t goes from 1 to 2 of F dot dr.

So in the first coordinates-- let me factor out the dt at the end. So that's going to be 3 t squared minus 2t, times 1, plus-- OK, well, let's expand this out now. 3t minus 2 quantity squared, that's going to give me a 9 t squared minus 12t plus 4-- so this is 9 t squared minus 12t plus 4, and then we have to add t squared to it. So this is plus 10 t squared minus 12t plus 4, times 3, and then this whole thing is dt.

dt is the whole integrand, there. I could even put in another pair of parentheses just to emphasize that, perhaps. OK. Now this is a straightforward-- I mean, it's a little complicated looking, but it's just an integral of a polynomial. Easy enough to do.

Let's first just combine terms. OK, so let's look at the t squareds. We have a 10 t squared times 3. So 30 t squared, and then another 3 times 1. So 33 t squared. And I've got minus 2t minus 36t is minus 38t, plus 12-- 4 times 3-- dt. OK, and now we integrate.

So this is equal to 11 t cubed-- that's a 3-- minus 19 t squared plus 12t as t varies between 1 and 2. And all right. OK, so now we've got to plug in and evaluate and so on. So at 2, this is 88 minus 76 plus 24, minus 11 minus 19 plus 12. So you do some arithmetic and this is going to

work out to 32.

OK, so there's part a. It's a nice, simple curve, so we had a nice, simple parametrization. We computed F and dr, then we dotted them, and integrated. OK, so now we're going to do the same exact thing for part b, but in part b, the curve is a little more complicated.

Let's come over here where we've got some empty space. So in part b, our curve looks like this. So it starts at the point (1, 1), and then it goes up to the point (1, 4), and then it goes over to the point (2, 4). All right? So it's hard to parametrize in one fell swoop something that makes a sharp right angle like that.

So a natural thing to do is to split the integral over this whole curve into the integrals over the two different pieces. So let's call this vertical part C_1 and this horizontal part C_2. And so we know that the integral over C of F dot dr is equal to the integral over C_1 of F dot dr plus the integral over C_2 of F dot dr. And so now, it's easy enough to parametrize these two separate curves separately.

C_1, for example, is the straight line segment that goes from (1, 1) to (1, 4). So C_1. So that means we have x equal to 1, and 1 less than or equal to y less than or equal to 4. So a natural parametrization here is just the parametrization that uses the parameter y. Right? So in this one, I'm not going to bother introducing a new letter t. I'm just going to stick with x and y. So we have x equals 1, and y is our parameter and it goes from 1 to 4.

So now let's look at what F and dr are. So in this case, F is equal to-- its first coordinate is x*y, and x is just 1 here, so this is y. And its second coordinate was x squared plus y squared, and so that's going to be 1 plus y squared.

And dr-- well, r here is 1 comma y-- so dr is equal to 0 comma dy. Or (0, 1) times dy, if you wanted to factor that dy out to the end. OK. Good. So we're all set to do that first integral. So let's do that.

So we have the integral over C_1 of F dot dr is equal to-- well, we dot these two things together. And the first term gives me y times 0, and that's just 0. So that's going to die, and all we're left with is the second term. So it's the integral of 1 plus y squared dy, but we need bounds. Right?

OK, so y was going from 1 to 4 in this integral. So it's the integral from 1 to 4 of 1 plus y

squared dy. OK. So we can either continue and evaluate this now, or we could go and do the second one. Let's finish evaluating it since we've already got it written up here.

So this is equal to y, plus y cubed over 3, between 1 and 4. So what is this? This is 4 plus 64/3, minus 1 plus 1/3. So that looks like it's 24 to me. OK, so we get 24 for the first part.

Now, let's do the second part. So $C_2$ here. I'll draw a little line there to separate them. Now, curve $C_2$-- let's go back and look at it-- OK, so curve $C_2$ is the segment connecting the points (1, 4) and the point (2, 4). OK, so y is always 4 on this curve, and x goes from 1 to 2. So 1 is less than or equal to x less than or equal to 2, y is equal to 4.

So a natural parametrization here, again, is just to take x to be our parameter. And again, I'm not going to introduce a letter t. We're just using x as our parameter. So in this case, F-- well, it's x*y, so x is just x and y is 4-- so that's 4x comma-- and the second coordinate is x squared plus y squared-- so that's x squared plus 16. And dr is equal to dx comma 0. OK, so that's F and dr.

So the integral that I want now is the integral over $C_2$ of F dot dr. OK, so we just plug in here what we've got. So this is equal to the integral of-- well, the first coordinates are 4x dx and the second coordinates just give me 0-- so it's 4x dx.

And again, I need my bounds. Well, I had-- over here, I had 1 less than or equal to x is less than or equal to 2. So that's the integral between 1 and 2. 4x-- integrate that-- and I get 2 x squared between 1 and 2, which is equal to 8 minus 2, or 6.

All right. So let's see what we've got. So we had-- back here, we had our curve C, which we split into the two parts $C_1$ and $C_2$. And we wanted to know what the integral over C was, and we've separately computed the integral over $C_1$. And we computed that to be 24. And we computed the integral over $C_2$, and that was 6. So the integral over the whole curve of F dot dr is equal to 24 plus 6, which is 30. OK. So there's your answer for the second part.

Now one thing I'd like you to notice is that over this curve C in part b-- over the whole curve in part b-- we got that the integral of this field F was 30. And now if you remember, right here, in the first part, in part a, we computed the integral over a different curve that connected the two same endpoints. And we found that the integral came out to 32.

So one thing you should take away from this is that the integral over a curve joining two points can depend on which curve you choose, right? So we had two different curves and we got two

different answers, even though the two curves connected the same points. So that's interesting.

And the other thing to take away from this is just the general approach. Which is that whenever you have a problem like this, what you want to do is you want to take your curve-- so whether it be-- well, in part a we had this straight line, slanted line. In part b where we had this nice piecewise linear with these vertical and horizontal parts-- you want to break it into nice pieces, parametrize them. You know, sometimes you only need one piece when it's an easy-to-parametrize curve like that. Sometimes, if it has corners or so on, you might want more pieces. Break it into pieces, choose a nice parametrization, and that reduces your problem just to computing integrals, just like we've done in Calculus I-- in 18.01-- and then you just integrate.

All right. I'll end there.