Fundamental Theorem for Line Integrals

1. Let $f = xy + \mathrm{e}^x$.

a) Compute $\mathbf{F} = \boldsymbol{\nabla} f$.

b) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for each of the following paths from (0,0) to (2,1).

i) The path consisting of a horizontal segment followed by a vertical segment.

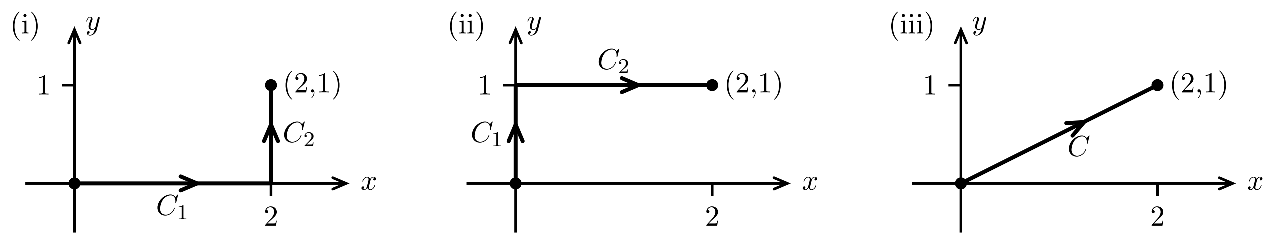
ii) The path consisting of a vertical segment followed by a horizontal segment.

iii) The straight line from (0,0) to (2,1).

c) All of the answers to part (b) should be the same. Show they agree with the answer given by the fundamental theorem for line integrals.

<u>**Answer:**</u> a) $\mathbf{F} = \boldsymbol{\nabla} f = \langle f_x, f_y \rangle = \langle y + \mathrm{e}^x, x \rangle$.

b) We have $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (y + \mathrm{e}^x)\,dx + x\,dy$.



i) The curve C has two pieces C_1 and C_2. We compute the integral over each piece separately.

C_1: x runs from 0 to 2; $y = 0$, $dy = 0$. So,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \mathrm{e}^x\,dx = \mathrm{e}^2 - 1.$$

C_2: $x = 2$, $dx = 0$; y runs from 0 to 1. So,

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 2\,dy = 2.$$

Summing the two pieces: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1+C_2} \mathbf{F} \cdot d\mathbf{r} = \mathrm{e}^2 + 1.$

ii) This is similar to part (i).

C_1: $x = 0$, $dx = 0$; y runs from 0 to 1. So,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 0\,dy = 0.$$

C_2: x runs from 0 to 2; $y = 1$, $dy = 0$. So,

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 1 + \mathrm{e}^x\,dx = 2 + \mathrm{e}^2 - 1 = 1 + \mathrm{e}^2.$$

Summing the two pieces: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1+C_2} \mathbf{F} \cdot d\mathbf{r} = \mathrm{e}^2 + 1.$

iii) We parametrize C by $x = 2t;\quad y = t;\quad t$ runs from 0 to 1 $\Rightarrow\ dx = 2\,dt,\ dy = dt$. Thus,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (t + \mathrm{e}^{2t})2\,dt + 2t\,dt = \int_0^1 (4t + 2\mathrm{e}^{2t})\,dt = 2t^2 + \mathrm{e}^{2t}\Big|_0^1 = 2 + \mathrm{e}^2 - 1 = 1 + \mathrm{e}^2.$$

c) The fundamental theorem for line integrals says (for any of the paths in part (b))

$$\int_C \boldsymbol{\nabla} f\cdot d\mathbf{r} = f(2,1) - f(0,0) = 2 + \mathrm{e}^2 - 1 = 1 + \mathrm{e}^2.$$

All the answers agree.