Problems: Green's Theorem

Calculate $\oint_C -x^2y\,dx + xy^2\,dy$, where C is the circle of radius 2 centered on the origin.

Answer: Green's theorem tells us that if $\mathbf{F} = \langle M, N \rangle$ and C is a positively oriented simple closed curve, then

$$\oint_C M\,dx + N\,dy = \iint_R N_x - M_y\,dA.$$

We let $M = -x^2y$ and $N = xy^2$ to get:

$$\begin{aligned}
\oint_C -x^2y\,dx + xy^2\,dy &= \iint_R y^2 - (-x^2)\,dA \\
&= \iint_R x^2 + y^2\,dA \\
&= \int_0^{2\pi}\int_0^2 r^2\,r\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{8}{3} d\theta \\
&= \frac{16\pi}{3}.
\end{aligned}$$

This result is 4/3 times the area $\iint_R 1\,dA$ of the circle, and so is a plausible answer.