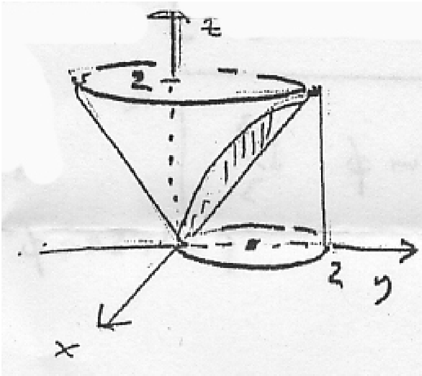
18.02 Problem Set 10, Part II Solutions

1. Base: R: $x^2 + (y-1)^2 \le 1$. Top: $z = f(x,y) = (x^2+y^2)^{1/2}$.



In cylindrical coords, base is

$$\begin{aligned} 0 \le \ & r & \le 2\sin\theta \\ 0 \le \ & \theta & \le \pi \end{aligned}$$

The top is $z = r$.

(a)

$$\begin{aligned} V &= \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r} 1dz\ rdr\ d\theta = \int_0^{\pi}\int_0^{2\sin\theta} r\cdot rdr\ d\theta \\ &= \int_0^{\pi} r^3/3|_0^{2\sin\theta} d\theta = \frac{8}{3}\int_0^{\pi}\sin^3\theta d\theta \\ &= \frac{16}{3}\int_0^{\pi/2}\sin^3\theta\ d\theta \end{aligned}$$

with the last step by symmetry. Then use table 113, $n = 3$ to get

$$= \frac{16}{3}\cdot\frac{2}{3} = \frac{32}{9}.$$

(b)

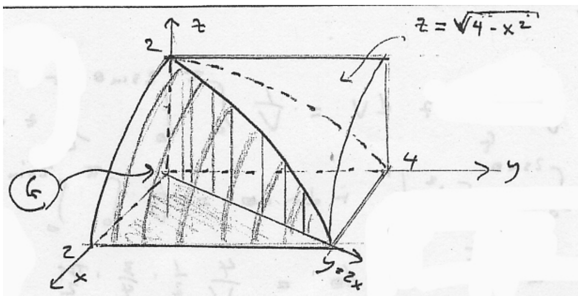
$$\begin{aligned}
\bar{z} &= \frac{1}{V}\int\int\int_G z dV = \frac{1}{V}\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r} z dz\ r dr\ d\theta \\
&= \frac{1}{V}\int_0^{\pi}\int_0^{2\sin\theta}\left[z^2/2\right]_0^r r\ dr\ d\theta \\
&= \frac{1}{V}\int_0^{\pi}\int_0^{2\sin\theta} r^3/2\ dr\ d\theta = \frac{2}{V}\int_0^{\pi}\sin^4\theta\ d\theta \\
&= \frac{4}{V}\int_0^{\pi/2}\sin^4\theta\ d\theta = \frac{4}{V}\frac{1}{2}\frac{3}{4}\frac{\pi}{2} = \frac{3\pi}{4V}.
\end{aligned}$$

Now $V = 32/9$ so $\bar{z} = \frac{27\pi}{128} \approx 0.6627$ and this is approximately a third of the maximum height (=2) of G.

2. Looking at the original integral, we see that G is the region

$$0 \le z \le \sqrt{4-x^2}, \quad 0 \le y \le 2x, \quad 0 \le x \le 2$$

(a)



(b) We may re-describe G as

$$0 \le y \le 2x, \quad 0 \le x \le \sqrt{4-z^2}, \quad 0 \le z \le 2$$

This gives

$$\int_0^2\int_0^{\sqrt{4-z^2}}\int_0^{2x} f(x,y,z)dy\ dx\ dz$$

(c) We may redescribe G as

$$y/2 \le x \le \sqrt{4-z^2}, \quad 0 \le z \le \sqrt{4-(y/2)^2}, \quad 0 \le y \le 4$$

This gives

$$\int_0^4 \int_0^{\sqrt{4-(y/z)^2}} \int_{y/2}^{\sqrt{4-z^2}} f(x,y,z)dx\ dz\ dy.$$

(d) $f = 1$. Then the original integral is

$$\begin{aligned}\int_0^2 \int_0^{2x} \int_0^{\sqrt{4-x^2}} dz\ dy\ dx &= \int_0^2 \int_0^{2x} (4-x^2)^{1/2} dy\ dx = \int_0^2 2x(4-x^2)^{1/2} dx \\ &= -\frac{2}{3}(4-x^2)^{3/2}\Big|_0^2 = 16/3.\end{aligned}$$

The integral from (b) is

$$\begin{aligned}\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{2x} dy\ dx\ dz &= \int_0^2 \int_0^{\sqrt{4-z^2}} 2x dx\ dz \\ &= \int_0^2 \left[x^2\right]_0^{(4-z^2)^{1/2}} dz = \int_0^2 (4-z^2)dz = \left[4z - z^3/3\right]_0^2 = 16/3.\end{aligned}$$

The integral from (c) is

$$\int_0^4 \int_0^{\sqrt{4-(y/z)^2}} \int_{y/2}^{\sqrt{4-z^2}} dx\, dz\, dy = \int_0^4 \int_0^{\sqrt{4-(y/2)^2}} (\sqrt{4-z^2} - y/2)\, dz\, dy =$$

$$= \int_0^4 \int_0^{\sqrt{4-(y/2)^2}} \sqrt{4-z^2}\ dz\, dy - \int_0^4 \int_0^{\sqrt{4-(y/2)^2}} y/2\ dz\, dy$$

Trig sub's and/or tables would be needed in order to get these to work out in exact form.

3. If V is the volume of the full vessel, we have that

$$\text{Energy} = \int\int\int_V dE$$

where

$$dE\ =\ g\, z\, dm\ =\ 10^3 g\, z\, dV$$

since the density of water is 10^3. Therefore we compute:

$$\text{Energy}\ = \int\int\int_V 10^3 g\, z\, dV = \int_0^{100} \int_{-9}^{9} \int_{\frac{1}{81}x^4}^{81} 10^3 g\, z\, dz\, dx\, dy\ =\ \ldots\ =\ 5.14{\cdot}10^{10}\,\text{joules}.$$