Welcome back to recitation. In this video I would like us to do the following problem. We're going to let z equal x squared plus y and we want S to be the graph of z above the unit square in the xy-plane. And what we'd like to do is for F equal to z*i plus x*k, find the upward flux of F through S. So S is our surface that's a graph, over the xy-plane and the unit square, of z equal to x squared plus y. And we want to compute the upward flux of F through that surface. So why don't you work on this problem, pause the video, and then when you're ready to see my solution, bring the video back up.

OK, welcome back. So again, what we want to do is we want to find the upward flux of F through this surface S. And let's think about first, how do we describe the surface S? S is the graph of z equal x squared plus y. So we can think of it as, z is really-- we can think of it as a function of x and y over the unit square. So we can say f of x, y is equal to z which is equal to x squared plus y.

And then we know how to compute the normal-- well, we know how to compute n*dS, which also in class is sometimes written notationally as dS with the vector dS. So we know how to compute this form right here. And it is-- you were shown in class that if f is-- or sorry. If you have a graph, if your surface is a graph, then this is exactly equal to the vector minus f sub x comma minus f sub y comma 1 dx dy. So that's exactly what this n*dS-- so n is the vector, and dS is the surface form we have here. So n*dS is exactly equal to the vector minus f sub x, minus f sub y, 1, dx dy.

So what do we have here with f sub x? f sub x-- because f is equal to z-- f sub x is 2x and f sub y is 1. So in our case we get exactly minus 2x, comma minus 1, comma 1, dx dy. And now to compute the surface integral what we do-- or sorry, to compute the flux along the surface, what we do is we integrate over the surface-- which I guess we should remember that's a double integral, because it's over a surface-- of F dotted with dS. But that's the same as integrating over the region.

So we have this surface, we know the region below that defines the surface in the xy-coordinates. So it's integrating over the region of F dotted with this vector here. Because n*dS-- dS is n*dS, and in the x-y components it's exactly equal to this. Minus 2x, minus 1, 1, dx dy. So now we're integrating. We've gone from looking at a surface integral. Now we're integrating-- we were integrating F dot dS on the surface, to now taking F dotted with this vector on the region in the xy-plane over which we can define S.

So the region we're interested in, remember, is the unit square. So we have the unit square which is x goes from 0 to 1 and y goes from 0 to 1. And then what we're doing is we're looking at F as a function of x, y, and z. And we want to dot that with this vector. And it's all being done in the variables x and y. So we should be able to change everything to x and y ultimately.

So let's look at what we get when we do that. So F-- I'm going to remind myself-- F was equal to $z*i$ plus $x*k$. Which, if I write that in the component form, it's z comma, 0 comma, x. So F dotted with our minus f sub x, minus f sub y, 1-- which was minus 2x, minus 1, 1-- we see we get-- minus 2x dotted with z-- we get minus 2x*z and then we get 0 and then we get x. So we get minus 2x*z plus x. That's exactly what f dotted with the vector we have is.

So now also, we know that z was equal to x squared plus y. So we actually get negative 2x times x squared plus y plus x again. So I'm going to just expand that so it's easier to deal with. So we get negative 2x cubed minus 2x*y plus x. And now we have exactly what-- if we look over here-- we have exactly this entire part here written as a function of x and y. Which is good. Why is that good? Because everything we're integrating is in x and y. We're doing dx and dy so we just need to figure out the bounds and compute the integral.

So let's come over here. So the flux then is going to be equal to-- well, we know the region. We know the region is y and x are both going from 0 to 1. So the order doesn't matter because nothing depends on another function. And then we're integrating exactly this function. Negative 2 x to the third minus 2x*y plus x dy dx.

So when we integrate in y, we should be careful, what do we get here? We're going to have the integral from 0 to 1, and then we're going to have-- this we get a negative 2 x cubed times y, and then evaluated at 0 and 1, so we just get a negative 2 x cubed again. We integrate this we have a negative 2x y squared over 2. So at 0 we get nothing and at 1 we get 1/2. And so we get minus 2x.

And then here when we integrate in y, we get x times y and we evaluate that at 1 and 0, and we got just plus x. So let me just make sure I didn't make any mistakes there. So this one, I'm integrating it in y and so I get a negative 2 x cubed y, evaluated at 0 and 1. So at 1 I just get a negative 2 x cubed, at 0 I get 0. In this one, I have a negative 2x*y. When I integrate that I get a y squared over 2. The 2s kill off. So I'm left with a negative of x y squared. Evaluating that at 0 and 1, at 0 I get 0 and at 1 I get negative x. Oh, there. So there shouldn't be a 2 there.

And then here, when I integrate that I get x*y evaluated at y equals 0 and y equals 1, and take that difference. And at 1 I get just x and at 0 I get nothing. Hopefully that one is correct now. Because I forgot to kill off the 2 there first. So those subtract off and I'm left with minus the integral from 0 to 1 of 2 x cubed dx. Well, that's going to be minus of x cubed, it's going to be x to the fourth over 4. And then I have the 2 still here. So that will divide out. Evaluate at 0 and 1. At 0 I obviously get nothing. At 1 I get negative 1/2.

And so the flux of F across the surface is equal to negative 1/2. And that's the upward flux. So obviously if I wanted to know the downward flux, that would be positive 1/2. It doesn't have anything to do with what F is. It has to do with the direction of the normal that I'm dotting F with. So since I was dotting F with the upward normal-- which is the n*dS that I showed you was the upward normal-- then I know that this is the upward flux.

So let me just remind you what we did here. Let's come back to the very beginning. So the object was that we had z as a function of x and y. So we knew we had a surface sitting over some region in the xy-plane. And we wanted to compute the flux of a certain vector field-- the upward flux of a certain vector field-- across that surface.

And so all we had to do to solve this problem was ultimately understand what n*dS was-- which you actually did in class. You saw what n*dS is, this is the upward normal through the surface. And then recognize that the flux-- again, we saw this from class-- that the flux is equal to the double integral over the surface of F dot dS, which is the same as the double integral over the region of F dotted with n*dS. Where n*dS, now, I'm referring to as-- n is the vector and dS is-- this whole component is n*dS-- that's what we found.

And so then we know F. It's in terms of z, x, and y. But then we can find it in terms of x and y. When we take that dot product we end up with exactly just a function of x and y, when we replace z by what it actually is equal to. And then we just compute the integral. And this is just a regular old double integral. And we get the flux was equal to minus 1/2. And again, I want to point out that if we wanted instead of the upward flux the downward flux, it would be the same with the opposite sign. OK. That is where I think I'll stop.