V12. Gradient Fields in Space

1. The criterion for gradient fields. The curl in space.

We seek now to generalize to space our earlier criterion (Section V2) for gradient fields in the plane.

Criterion for a Gradient Field. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ be continuously differentiable. Then

$$(1)\quad \mathbf{F} = \nabla f \quad \text{for some } f(x,y,z) \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\,, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}\,, \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}\,.$$

Proof. Since $\mathbf{F} = \nabla f$, when written out this says

$$(2)\qquad M = \frac{\partial f}{\partial x}, \qquad N = \frac{\partial f}{\partial y}. \qquad P = \frac{\partial f}{\partial z}\,; \qquad \text{therefore}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}\,.$$

The two mixed partial derivatives are equal since they are continuous, by the hypothesis that $\mathbf{F}$ is continuously differentiable.

The other two equalities in (1) are proved similarly. □

Though the criterion looks more complicated to remember and to check than the one in two dimensions, which involves just a single equation, it is not difficult to learn and apply. For theoretical purposes, it can be expressed more elegantly by using the three-dimensional vector curl $\mathbf{F}$.

Definition. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ be differentiable. We define **curl F** by

$$(3)\qquad \operatorname{curl} \mathbf{F} = (P_y - N_z)\,\mathbf{i} + (M_z - P_x)\,\mathbf{j} + (N_x - M_y)\,\mathbf{k}$$

$$(3')\qquad = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} \qquad (\text{symbolic notation; } \partial_x = \frac{\partial}{\partial x}, \text{ etc.})$$

$$(3'')\qquad = \nabla \times \mathbf{F}, \qquad \text{where } \nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}\,.$$

The equation (3) is the definition. The other two lines give symbolic ways of writing and of remembering the right side of (3). Neither the first nor second row of the determinant contains the sort of thing you are allowed to put into a determinant; however, if you "evaluate" it using the Laplace expansion by the first row, what you get is the right side of (3). Similarly, to evaluate the symbolic cross-product in (3″), we use the determinant (3′). In doing these, by the "product" of $\dfrac{\partial}{\partial x}$ and M we mean $\dfrac{\partial M}{\partial x}$.

By using the vector field curl $\mathbf{F}$, our criterion (1) becomes

$$(1')\qquad \mathbf{F} = \nabla f \quad \Rightarrow \quad \operatorname{curl} \mathbf{F} = \mathbf{0}\,.$$

In dealing with a plane vector field $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$, we gave the name curl $\mathbf{F}$ to the *scalar* function $N_x - M_y$, whereas for a vector field $\mathbf{F}$ in space, curl $\mathbf{F}$ is a *vector* function. However, if we think of the two-dimensional field $\mathbf{F}$ as a field in space (i.e., one with zero $\mathbf{k}$-component and not depending on z), then using definition (3) you can compute that

$$\text{curl}\,\mathbf{F} \;=\; (N_x - M_y)\,\mathbf{k}\ .$$

Thus curl $\mathbf{F}$ has only a $\mathbf{k}$-component, so if we are dealing just with two-dimensional fields, it is natural to give the name curl $\mathbf{F}$ just to this $\mathbf{k}$-component. This is not a universally accepted terminology, however; some call it the "scalar curl", others don't use any name at all for $N_x - M_y$.

Naturally, the question arises as to whether the converse of ($1'$) is true — if curl $\mathbf{F} = \mathbf{0}$, is $\mathbf{F}$ a gradient field? As in two dimensionas, this requires some sort of restriction on the domain, and we will return to this point after we have studied Stokes' theorem. For now we will assume the domain is the whole three-space, in which case it is true:

Theorem. *If* $\mathbf{F}$ *is continuously differentiable for all* x, y, z,

$$\text{(4)}\qquad \textit{curl}\,\mathbf{F} \;=\mathbf{0} \quad\Rightarrow\quad \mathbf{F} \;=\; \nabla f, \qquad \textit{for some differentiable } f(x,y,z).$$

We will prove this later. If $\mathbf{F}$ is a gradient field, we can calculate the corresponding (mathematical) potential function $f(x,y,z)$ by the three-dimensional analogue of either of the two methods described before (Section V2). We illustrate with an example.

Example 1. For what value(s), if any, of c will $\mathbf{F} \;=\; y\,\mathbf{i} + (x + cyz)\,\mathbf{j} + (y^2 + z^2)\,\mathbf{k}$ be a conservative (i.e., gradient) field? For each such c, find a corresponding potential function $f(x,y,z)$.

Solution. Using (1) and (4), we calculate the relevant partial derivatives:

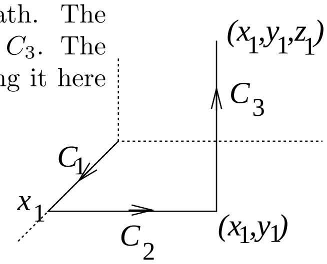
$$M_y = 1,\ N_x = 1; \quad N_z = cy,\ P_y = 2y; \quad M_z = 0,\ P_x = 0$$

Thus all three equations in (1) are satisfied $\Leftrightarrow$ $c = 2$. For this value of c, we now find $f(x,y,z)$ by two methods.

Method 1. We use the second fundamental theorem (Section V11, (11)), taking $(0,0,0)$ as a convenient lower limit for the integral, and using the subscript 1 on the upper limit to avoid confusion with the variables of integration. This gives

$$\text{(5)}\qquad f(x_1,y_1,z_1) \;=\; \int_{(0,0,0)}^{(x_1,y_1,z_1)} y\,dx + (x + 2yz)\,dy + (y^2 + z^2)\,dz$$

Since the integral is path-independent for the choice $c = 2$, we can use any path. The usual choice is the path illustrated, consisting of three line segments C_1, C_2 and C_3. The parametrizations for them are (don't write these out yourself — we are only doing it here this first time to make it clear how the line integral is being calculated):



$$\begin{array}{llll} C_1: & x = x,\ y = 0,\ z = 0; & \text{thus} & dx = dx,\ dy = 0,\ dz = 0; \\ C_2: & x = x_1,\ y = y,\ z = 0; & \text{thus} & dx = 0,\ dy = dy,\ dz = 0; \\ C_3: & x = x_1,\ y = y_1,\ z = z; & \text{thus} & dx = 0,\ dy = 0,\ dz = dz\ . \end{array}$$

Using these, we calculate the line integral (5) over each of the C_i in turn:

$$\begin{aligned}\int_{C_1+C_2+C_3} y\,dx + (x+2yz)\,dy + (y^2+z^2)\,dz \;&=\; \int_0^{x_1} 0\cdot dx \;+\; \int_0^{y_1} (x_1 + 2y\cdot 0)\,dy \;+\; \int_0^{z_1} (y_1^2+z^2)\,dz \\ &=\; 0 + x_1 y_1 + (y_1^2 z_1 + \frac{1}{3} z_1^3)\ .\end{aligned}$$

Dropping subscripts, we have therefore by (5),

$$f(x,y,z) \;=\; xy + y^2 z + \frac{1}{3}z^3 + c, \tag{6}$$

where we have added an arbitrary constant of integration to compensate for our arbitary choice of $(0,0,0)$ as the lower limit of integration — a different choice would have added a constant to the right side of (6).

The work should always be checked; from (6) one sees easily that $\nabla f \;=\; \mathbf{F}$, the field we started with.

Method 2. This requires no line integrals, but the work must be carried out systematically, otherwise you'll get lost in a mess of equations.

We are looking for an $f(x,y,z)$ such that $(f_x, f_y, f_z) \;=\; (y,\ x+2yz,\ y^2+z^2)$. This is equivalent to the three equations

$$f_x \;=\; y, \qquad f_y \;=\; x+2yz, \qquad f_z \;=\; y^2+z^2\ . \tag{7}$$

From the first equation, integrating with respect to x (holding y and z fixed), we get

$$\begin{aligned}
&(8) & f(x,y,z) &= xy + g(y,z), && g \text{ is an arbitrary function}\\
& & \frac{\partial f}{\partial y} &= x + \frac{\partial g}{\partial y}, && \text{from (8)}\\
& & &= x + 2yz && \text{from (7), second equation; comparing,}\\
& & \frac{\partial g}{\partial y} &= 2yz. && \text{Integrating with respect to } y,\\
& & g(y,z) &= y^2 z + h(z), && h \text{ is an arbitrary function; thus}\\
&(9) & f(x,y,z) &= xy + y^2 z + h(z), && \text{from the preceding and (8)}\\
& & \frac{\partial f}{\partial z} &= y^2 + h'(z) && \\
& & &= y^2 + z^2, && \text{from (7), third equation; comparing,}\\
& & h'(z) &= z^2, && \\
& & h(z) &= \frac{1}{3}z^3 + c; && \text{finally, by (9)}\\
& & f(x,y,z) &= xy + y^2 z + \frac{1}{3}z^3 + c && \text{as in Method 1.}
\end{aligned}$$

2. Exact differentials

Just as we did in the two-dimensional case, we translate the previous ideas into the language of differentials.

The formal expression

$$M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz \tag{10}$$

which appears as the integrand in our line integrals is called a **differential**. If $f(x,y,z)$ is a differentiable function, then its **total differential** (or just *differential*) is defined to be

$$df \;=\; \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz \tag{11}$$

The differential (10) is said to be **exact**, in some domain D where M, N and P are defined, if it is the total differential of some differentiable function $f(x,y,z)$ in this domain, that is, if there exists an $f(x,y,z)$ in D such that

$$M \;=\; \frac{\partial f}{\partial x}, \quad N \;=\; \frac{\partial f}{\partial y}, \quad P \;=\; \frac{\partial f}{\partial z}\;. \tag{12}$$

Criterion for exact differentials. *Let D be a domain in which M, N, P are continuously differentiable. Then in D,*

$$M\,dx + N\,dy + P\,dz \quad \text{is exact} \quad \Rightarrow \quad P_y = N_z, \quad M_z = P_x, \quad N_x = M_y\;; \tag{13}$$

if D is all of 3-space, then the converse is true:

$$P_y = N_z, \quad M_z = P_x, \quad N_x = M_y \quad \Rightarrow \quad M\,dx + N\,dy + P\,dz \quad \text{is exact.} \tag{14}$$

If the test in this criterion shows that the differential (10) is exact, the function $f(x,y,z)$ may be found be either method 1 or method 2. The converse (14) is true under weaker hypotheses about D, which we will come back to after we have taken up Stokes' Theorem.