Problems: Curl in 3D

1. Let $\mathbf{F} = \langle x, y, z \rangle$. Calculate and interpret $\text{curl}\mathbf{F}$.

<u>**Answer:**</u>

$$\begin{aligned}
\text{curl}\mathbf{F} &= \boldsymbol{\nabla} \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\
&= \langle 0, 0, 0 \rangle.
\end{aligned}$$

We can interpret $\mathbf{F}$ as a velocity field in which particles race away from $(0, 0, 0)$ with speed equal to their distance from the origin. Because $\text{curl}\mathbf{F} = \mathbf{0}$, there is no rotational component in this field.

This makes sense. If we place a tiny paddle wheel at the point $(1, 0, 0)$, the torque about the axis from the force on a paddle just below the x-axis is exactly cancelled by the one symmetrically placed just above the x-axis. Since all the forces all the way around the wheel come in such pairs the net torque is 0 and the wheel will not start to spin. A similar argument holds for all other points in the plane except the origin. At the origin there is no force on any of the paddles, so the wheel doesn't spin.

2. Let $\mathbf{F} = \langle y, 0, 0 \rangle$. Calculate and interpret $\text{curl}\mathbf{F}$.

<u>**Answer:**</u>

$$\begin{aligned}
\text{curl}\mathbf{F} &= \boldsymbol{\nabla} \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} \\
&= \langle 0, 0, -1 \rangle.
\end{aligned}$$

We can interpret $\mathbf{F}$ as a velocity field in which particles move away from the yz-plane with speed equal to their y coordinate. A tiny paddle wheel placed in this field with its axle pointing in the $\mathbf{k}$ direction will spin clockwise. Applying the right hand rule confirms the calculation that this field has a negative rotational component.

A paddle wheel placed with its axle pointing in the $\mathbf{i}$ direction will be completely unaffected by the velocity field, while the forces on the paddles of a wheel with its axle pointing in the $\mathbf{j}$ direction will cancel, as in problem 1.