

18.03 Class 22, March 31, 2010

### Fourier Series III

- [1] Differentiation and integration
- [2] Harmonic oscillator with periodic input
- [3] What you hear

[1] You can differentiate and integrate Fourier series.

Example: Consider the function  $f(t)$  which is periodic of period  $2\pi$  and is given by  $f(t) = |t|$  between  $-\pi$  and  $\pi$ .

We could calculate the coefficients, using the fact that  $f(t)$  is even and integration by parts. For a start,  $a_0/2$  is the average value, which is  $\pi/2$ .

Or we could realize that

$$f'(t) = \text{sgn}(t) \quad (\text{except where } f'(t) \text{ doesn't exist})$$

or what is the same

$$f(t) = \int_0^t \text{sgn}(u) \, du$$

and integrate the Fourier series of the squarewave.

NB: it is not true in general that the integral of a periodic function is periodic; think of integrating the constant function 1 for example. But the integral IS periodic if the average value of the function is zero. If you think of this one term at a time, the point is that the integral of  $\cos(nt)$  is periodic unless  $n = 0$  and the integral of  $\sin(nt)$  is always periodic.

Let's compute:

$$\begin{aligned} f(t) &= \int_0^t (4/\pi) \sum_{\{n \text{ odd}\}} \sin(nx)/n \, dx \\ &= (4/\pi) \sum_{\{n \text{ odd}\}} \int_0^t \sin(nx)/n \, dx \\ &= (4/\pi) \sum_{\{n \text{ odd}\}} [-\cos(nx) / n^2]_0^t \\ &= (4/\pi) \sum_{\{n \text{ odd}\}} (1/n^2) - (4/\pi) \sum_{\{n \text{ odd}\}} \cos(nt) / n^2 \end{aligned}$$

That's it, that's the Fourier series for  $f(t)$ . The constant term is a little odd. It's a specific number, but not a sum you can find by the geometric series or by telescoping. In fact the only way to evaluate it is this way, using Fourier series. Because we know that the constant term in the Fourier series for  $f(t)$  is the average value of  $f(t)$ , which is  $\pi/2$ :

$$\begin{aligned} (4/\pi) \sum_{\{n \text{ odd}\}} (1/n^2) &= \pi/2 \quad \text{or} \\ \sum_{\{n \text{ odd}\}} (1/n^2) &= \pi^2 / 8 . \end{aligned}$$

That is,

$$(\text{odd})^2: \quad 1 + 1/3^2 + 1/5^2 + 1/7^2 + \dots = \pi^2 / 8.$$

Just to carry this one step further: Try to sum all the reciprocal squares.

$$(2 \times \text{odd})^2: \quad 1/2^2 + 1/6^2 + 1/(10)^2 + \dots = (1/4) \pi^2 / 8$$

$$(4 \times \text{odd})^2: \quad 1/4^2 + 1/(12)^2 + 1/(20)^2 + \dots = (1/4)^2 \pi^2 / 8$$

...

$$\text{so} \quad \sum 1/n^2 = ( 1 + 1/4 + (1/4)^2 + (1/4)^3 + \dots ) \pi^2 / 8$$

The first factor is a geometric series:

$$1 + 1/4 + (1/4)^2 + \dots = 1/(1 - (1/4)) = 4/3 \quad \text{so}$$

$$1 + 1/2^2 + 1/3^2 + 1/4^2 + \dots = \pi^2/6$$

This is one of the most famous equations in all of mathematics.

It made Euler's reputation when he discovered it in 1736.

So we learn two things from this calculation: this interesting mathematical formula, and the calculation of the Fourier series for  $|t|$  extended periodically:

$$f(t) = \pi/2 - (4/\pi) \sum_{\{n \text{ odd}\}} \cos(nt) / n^2 .$$

[2] Harmonic oscillator with periodic forcing.

Now we come to the relationship with differential equations:

We have a complicated wave, perhaps a square wave,  $f(t)$  .

We drive a harmonic oscillator with it:

$$x'' + \omega_n^2 x = f(t)$$

What is the system response? We might imagine the system as a radio tuner;  $f(t)$  represents the radio wave, and  $x$  represents the output of the receiver.

Remember [Slide]:  $x'' + \omega_n^2 x = \cos(\omega t)$

has sinusoidal solution  $x_p = A \cos(\omega t) / (\omega_n^2 - \omega^2)$

and  $x'' + \omega_n^2 x = \sin(\omega t)$

has sinusoidal solution  $x_p = A \sin(\omega t) / (\omega_n^2 - \omega^2)$

When the denominator vanishes we have resonance and no periodic solution.

Well, by Superposition III we can now handle ANY periodic input signal. For example, suppose

$$f(t) = \text{sq}(t) = (4/\pi) (\sin(t) + (1/3) \sin(3t) + \dots )$$

Then we will have a particular solution

$$x_p = (4/\pi) (\sin(t)/(\omega_n^2 - 1) + \sin(3t)/(\omega_n^2 - 9) + \dots)$$

I showed the Harmonic Frequency Response applet. This applet actually shows the system response of a spring system driven through the spring, so it is

$$x'' + \omega_n^2 x = \omega_n^2 f(t)$$

so when  $f(t) = \sin(t)$ ,  $x_n = \omega_n^2 \sin(t) / (\omega_n^2 - 1)$

with amplitude  $\omega_n^2 / (\omega_n^2 - 1)$ . This is the "RMS" graph.

When  $f(t) = \sin(kt)$ ,

$$x_p = \omega_n^2 (4/\pi) (\sin(t)/(\omega_n^2 - 1) + \sin(3t)/(\omega_n^2 - 3^2) + \dots)$$

There is resonance when

$$\omega_n = 1, 3, 5, \dots$$

but NOT when  $\omega_n = 2, 4, 6, \dots$

When  $\omega_n$  is very near to but less than  $k^2$ ,  $k$  odd, the term

$$\sin(kt)/(\omega_n^2 - k^2)$$

is a large negative multiple of  $\sin(kt)$ . This appears on the applet.

Then when  $\omega_n$  passes  $k^2$  the dominant term flips sign and becomes a large positive multiple of  $\sin(kt)$ .

[3] Let's write  $\omega = \pi/L$  for the fundamental circular frequency of the periodic function  $f(t)$ . The Fourier series

$$f(t) = a_0/2 + a_1 \cos(\omega t) + a_2 \cos(2\omega t) + \dots \\ + b_1 \sin(\omega t) + b_2 \sin(2\omega t) + \dots$$

can be rewritten in polar form as

$$f(t) = A_0 + A_1 \cos(\omega t - \phi_1) + A_2 \cos(2\omega t - \phi_2) + \dots$$

If you think of this as the pressure variation at your eardrum, the  $A_0$  is atmospheric pressure. What you hear is the rest.

I showed the Fourier Coefficients: Complex with Sound Mathlet.

Notice how dramatically the phase alters the waveform.

It turns out that your ear hears only the amplitudes of the various Fourier components, or harmonics, not their relative phases. Just listen ....

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