

18.03 Class 23, April 2, 2010

Step and delta

- [1] Step function $u(t)$
- [2] Rates and $\delta(t)$
- [3] Regular, singular, and generalized functions
- [4] Generalized derivative
- [5] Heaviside and Dirac

Two additions to your mathematical modeling toolkit.

- Step functions [Heaviside]
- Delta functions [Dirac]

[1] Model of on/off process: a light turns on; first it is dark, then it is light. The basic model is the Heaviside unit step function

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

Of course a light doesn't reach its steady state instantaneously; it takes a small amount of time. If we use a finer time scale, you can see what happens. It might move up smoothly; it might overshoot; it might move up in fits and starts as different elements come on line. At the longer time scale, we don't care about these details. Modeling the process by $u(t)$ lets us just ignore those details. We simplify the model by supposing that $u(t) = 0$ for all $t < 0$ no matter how near to zero, $u(t) = 1$ for all $t > 0$ no matter how near to zero, and $u(0)$ is left undefined.

$u(t-a)$ turns on at $t = a$.

If $a < b$, $u(t-a) - u(t-b)$ turns on at $t = a$ and off again at $t = b$: it's a "window."

We can use $u(t-a)$ to turn on another function:

$u(t-a)f(t)$ is zero when $t < a$ and agrees with $f(t)$ when $t > a$.

Q1: What is the equation for the function which agrees with $f(t)$ between a and b ($a < b$) and is zero outside this window?

- (1) $(u(t-b) - u(t-a)) f(t)$
- (2) $(u(t-a) - u(t-b)) f(t-a)$
- (3) $(u(t-a) - u(t-b)) f(t)$
- (4) $u(t-a) f(t-a) - u(t-b) f(t-b)$
- (5) none of these

Ans: (3).

[2] From bank accounts to delta functions.

Bank account equation: $x' + Ix = q(t)$
 $x = x(t) = \text{balance} \quad (\text{K}\$)$
 $I = \text{interest rate} \quad ((\text{yr})^{-1})$
 $q(t) = \text{rate of savings} \quad (\text{K}\$/\text{yr})$

I am happily saving at the rate $\text{K}\$1/\text{yr}$. The concept of rate can be clarified] by thinking about the cumulative total, $Q(t)$ (from some starting time, perhaps $t = 0$);

$$Q'(t) = q(t)$$

or $Q(t) = \int_0^t q(u) du$

At $t = 1$ I won $\$2\text{K}$ at the race track! I deposit this into the account. I can model the cumulative total deposit using the step function:

$$Q(t) = t + 2 u(t-1)$$

What about the rate? For this we would need to be able to talk about the derivative of $u(t)$, in such a way that its integral recovers $u(t)$.

Of course there is no such function.

But let's approximate $u(t)$ as we did before. I can differentiate each of them, to form a rate (varying with time).

Once again, I don't care about the details. What matters is the two properties:

- The integral is always 1
- Concentrated very near to zero.

These functions approximate *something*, written $\delta(t)$; and

$$\delta(t) = u'(t)$$

Using this we can write down a formula for the new rate of savings:

$$q(t) = 1 + 2 \delta(t-1)$$

We can graph this using a "harpoon" at $t = 1$ with the number 2 next to it; the area under the harpoon is 2.

Not too long after this, at $t = 2$, I bought a camera for $\$3\text{K}$.

$$q(t) = 1 + 2 \delta(t-1) - 3 \delta(t-2) .$$

The negative multiple of δ can be represented using a harpoon pointing up with -3 next to it, or by a harpoon pointing down with $+3$ next to it.

[3] Let me go back and be more precise about the kind of functions we have been using before these δ things.

A function is "piecewise smooth" if it is broken up into segments so that:
- each segment has all higher derivatives
- at each endpoint, left and right limits of all derivatives exist.

We'll call piecewise smooth functions "regular." We can now add in combinations of delta functions, called "singularity functions." A sum of a regular function and a linear combination of delta functions is a "generalized function":

$$f(t) = f_r(t) + f_s(t)$$

For example, the $q(t)$ above has

$$q_r(t) = u(t)$$

$$q_s(t) = 2 \delta(t-1) - 3 \delta(t-2)$$

I can differentiate a regular function, but if there are breaks in the graph the result will be a generalized function:

Whenever $Q(a+)$ is different from $Q(a-)$, the derivative will have a delta contribution:

$$(Q(a+) - Q(a-)) \delta(t-a)$$

Keeping these terms in the derivative lets us reconstruct $f(t)$ up to a constant. With the singular terms in place this is called the "generalized derivative."

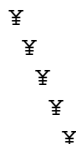
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*****
*                                     *
*   In this Unit whenever we differentiate   *
*   a discontinuous function we will mean   *
*   the generalized derivative.             *
*                                     *
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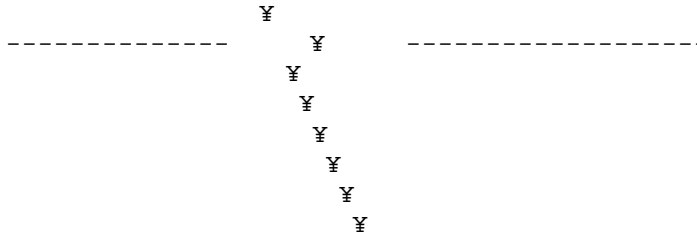
[4] When you fire a gun, you exert a very large force on the bullet over a very short period of time. If we integrate $F = ma = mx''$ we see that a large force over a short time creates a sudden change in the momentum, mx' . This is called an "impulse."

I fire straight up. The graph of the elevation of the bullet, plotted against t , starts at zero, then abruptly rises in an inverted parabola, and then when it hits the ground it stops again.

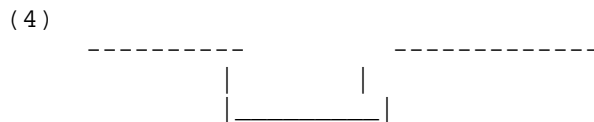
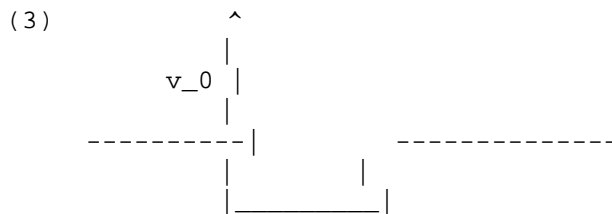
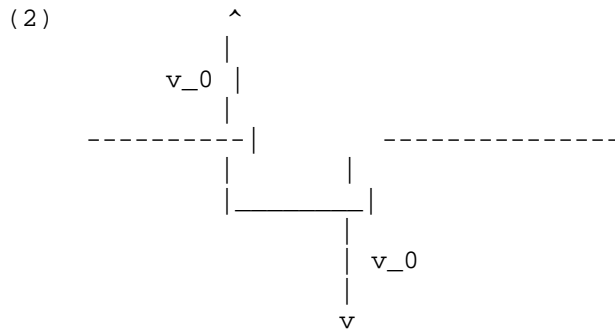
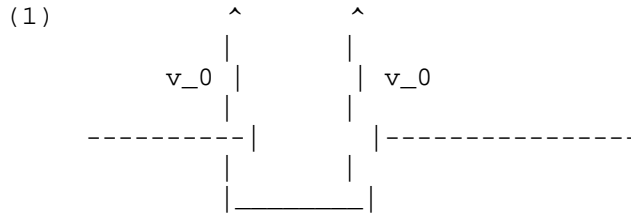
The derivative is zero for $t < 0$; then it rises abruptly to v_0 ; then it falls at constant slope (the acceleration of gravity) till the instant when it hits the ground, when it returns abruptly to zero.

$v(t)$ has a graph like this:





Q2: What does the graph of the generalized derivative of $v(t)$ look like?



Ans: (1).

[5] Oliver Heaviside, 1850--1925, British mathematical engineer
 ``... whose profound researches into electro-magnetic waves
 have penetrated further than anyone yet understands.''

He was the one who wrote down Maxwell's equations in the compact vector
 form you see now on ``Let there be light'' T-shirts.]

Paul A. M. Dirac, 1902--1984, Swiss/British theoretical physicist.
Nobel prize 1933, for the relativistic theory of the electron.

Lucasian Chair, Cambridge University:

Isaac Newton, 1669

...

P.A.M. Dirac, 1932--1969

...

Stephen Hawking, 1980

Quotes:

``I consider that I understand an equation when I can predict
the properties of its solutions without actually solving it.''
(Quoted in Frank Wilczek and Betsy Devine, "Longing for the Harmonies")

``God used beautiful mathematics in creating the world.'']

[6: Supplement] People often want to know what the delta function REALLY IS.
One answer is that it is a symbol, representing a certain approximation
to reality and obeying certain rules.

There are other answers.

One is this: one measures the value of a function by means of a piece
of equipment of some sort. This equipment gathers light, for example, over
a period of time, and reports an integrated value. The time interval may
be short but it is not of width zero. Each measuring device has a
sensitivity profile, $m(t)$, which rises to a peak and then falls again,
and which mathematically is assumed to be smooth.

If the light profile is $f(t)$, what this instrument actually measures is

$$M(f;m) = \text{integral } f(t) m(t) dt$$

The most we can ever know about the function $f(t)$ is the collection of
all these measurements, $M(f;m)$ as m varies over all measuring devices.

So f determines a new "function," sending each m to a number.
This is what is called a "distribution." Notice that changing the value
of $f(t)$ at a single point doesn't change $M(f;m)$: the distribution
determined by a function is independent of the value of the function at
any single point.

There are other ways to assign a number to each measuring device.
For example, we can send m to $m(0)$. That is what the "delta function"
does:

$$\text{integral } \delta(t) m(t) dt = m(0)$$

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