18.03 Class 23, April 2, 2010

Step and delta

[1] Step function u(t)
[2] Rates and delta(t)
[3] Regular, singular, and generalized functions
[4] Generalized derivative
[5] Heaviside and Dirac

Two additions to your mathematical modeling toolkit.
- Step functions [Heaviside]
- Delta functions [Dirac]

[1] Model of on/off process: a light turns on; first it is dark, then it is light. The basic model is the Heaviside unit step function

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    u(t) = 0        for  t < 0
           1        for  t > 0
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Of course a light doesn't reach its steady state instantaneously; it takes a small amount of time. If we use a finer time scale, you can see what happens. It might move up smoothly; it might overshoot; it might move up in fits and starts as different elements come on line. At the longer time scale, we don't care about these details. Modeling the process by u(t) lets us just ignore those details. We simplify the model by supposing that u(t) = 0 for all t < 0 no matter how near to zero,
u(t) = 1 for all t > 9 no matter how near to zero,
and u(0) is left undefined.

u(t-a) turns on at t = a .

If a < b , u(t-a) - u(t-b) turns on at t = a and off again at t = b : it's a "window."

We can use u(t-a) to turn on another function:
u(t-a) f(t) is zero when t < a and agrees with f(t) when t > a .

Q1: What is the equation for the function which agrees with f(t) between a and b (a < b) and is zero outside this window?

(1) (u(t-b) - u(t-a)) f(t)
(2) (u(t-a) - u(t-b)) f(t-a)
(3) (u(t-a) - u(t-b)) f(t)
(4) u(t-a) f(t-a) - u(t-b) f(t-b)
(5) none of these

Ans: (3).

[2] From bank accounts to delta functions.

Bank account equation: $x' + Ix = q(t)$
$x = x(t)$ = balance (K$)
I = interest rate ($(yr)^{-1}$)
$q(t)$ = rate of savings (K$/yr)

I am happily saving at the rate K$1/yr. The concept of rate can be clarified] by thinking about the cumulative total, $Q(t)$ (from some starting time, perhaps $t = 0$);

$$Q'(t) = q(t)$$

or $$Q(t) = \int_0^t q(u)\,du$$

At $t = 1$ I won $2K at the race track! I deposit this into the account. I can model the cumulative total deposit using the step function:

$$Q(t) = t + 2\,u(t-1)$$

What about the rate? For this we would need to be able to talk about the derivative of $u(t)$, in such a way that its integral recovers $u(t)$.

Of course there is no such function.

But let's approximate $u(t)$ as we did before. I can differentiate each of them, to form a rate (varying with time).

Once again, I don't care about the details. What matters is the two properties:

- The integral is always 1
- Concentrated very near to zero.

These functions approximate *something*, written $\delta(t)$; and

$$\delta(t) = u'(t)$$

Using this we can write down a formula for the new rate of savings:

$$q(t) = 1 + 2\,\delta(t-1)$$

We can graph this using a "harpoon" at $t = 1$ with the number 2 next to it; the area under the harpoon is 2.

Not too long after this, at $t = 2$, I bought a camera for $3K.

$$q(t) = 1 + 2\,\delta(t-1) - 3\,\delta(t-2) .$$

The negative multiple of delta can be represented using a harpoon pointing up with -3 next to it, or by a harpoon pointing down with $+3$ next to it.

[3] Let me go back and be more precise about the kind of functions we have been using before these delta things.

A function is "piecewise smooth" if it is broken up into segments so that:
- each segment has all higher derivatives
- at each endpoint, left and right limits of all derivatives exist.

We'll call piecewise smooth functions "regular." We can now
add in combinations of delta functions, called "singularity functions."
A sum of a regular function and a linear combination of delta functions
is a "generalized function":

$$f(t) = f_r(t) + f_s(t)$$

For example, the $q(t)$ above has

$$q_r(t) = u(t)$$

$$q_s(t) = 2 \delta(t-1) - 3 \delta(t-2)$$

I can differentiate a regular function, but if there are breaks in
the graph the result will be a generalized function:
Whenever $Q(a+)$ is different from $Q(a-)$,
the derivative will have a delta contribution:

$$(Q(a+) - Q(a-)) \delta(t-a)$$

Keeping these terms in the derivative lets us reconstruct $f(t)$ up to a
constant. With the singular terms in place this is called the
"generalized derivative."

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**********************************************
*                                      *
*    In this Unit whenever we differentiate   *
*    a discontinuous function we will mean    *
*    the generalized derivative.          *
*                                      *
**********************************************
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[4] When you fire a gun, you exert a very large force on the bullet
over a very short period of time. If we integrate $F = ma = mx''$
we see that a large force over a short time creates a sudden change in
the momentum, mx' . This is called an "impulse."

I fire straight up.
The graph of the elevation of the bullet, plotted against t,
starts at zero, then abruptly rises in an inverted parabola, and then
when it hits the ground it stops again.

The derivative is zero for $t < 0$; then it rises abruptly to v_0;
then it falls at constant slope (the acceleration of gravity) till
the instant when it hits the ground, when it returns abruptly to zero.

$v(t)$ has a graph like this:

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                 ¥
                  ¥
                   ¥
                    ¥
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                        ¥
                         ¥
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```

Q2: What does the graph of the generalized derivative of v(t) look like?

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(1)           ^       ^
              |       |
          v_0 |       | v_0
              |       |
     ----------|       |----------------
              |       |
              |_______|


(2)           ^
              |
          v_0 |
              |
     ----------|        ---------------
              |       |
              |_______|
                      |
                      | v_0
                      |
                      v

(3)           ^
              |
          v_0 |
              |
     ----------|         --------------
              |        |
              |________|


(4)
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              |        |
              |________|
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Ans: (1).

[5] Oliver Heaviside, 1850--1925, British mathematical engineer
``... whose profound researches into electro-magnetic waves
have penetrated further than anyone yet understands.''

He was the one who wrote down Maxwell's equations in the compact vector
form you see now on ``Let there be light'' T-shirts.]

Paul A. M. Dirac, 1902--1984, Swiss/British theoretical physicist.
Nobel prize 1933, for the relativistic theory of the electron.

Lucasian Chair, Cambridge University:

Isaac Newton, 1669
...
P.A.M. Dirac, 1932--1969
...
Stephen Hawking, 1980

Quotes:

``I consider that I understand an equation when I can predict the properties of its solutions without actually solving it.''
(Quoted in Frank Wilczek and Betsy Devine, "Longing for the Harmonies")

``God used beautiful mathematics in creating the world.'']

[6: Supplement] People often want to know what the delta function REALLY IS. One answer is that it is a symbol, representing a certain approximation to reality and obeying certain rules.

There are other answers.

One is this: one measures the value of a function by means of a piece of equipment of some sort. This equipment gathers light, for example, over a period of time, and reports an integrated value. The time interval may be short but it is not of width zero. Each measuring device has a sensitivity profile, m(t) , which rises to a peak and then falls again, and which mathematically is assumed to be smooth.

If the light profile is f(t) , what this instrument actually measures is

M(f;m) = integral f(t) m(t) dt

The most we can ever know about the function f(t) is the collection of all these measurements, M(f;m) as m varies over all measuring devices.

So f determines a new "function," sending each m to a number.
This is what is called a "distribution." Notice that changing the value of f(t) at a single point doesn't change M(f;m) : the distribution determined by a function is independent of the value of the function at any single point.

There are other ways to assign a number to each measuring device.
For example, we can send m to m(0). That is what the "delta function" does:

integral delta(t) m(t) dt = m(0)