

18.03 Class 37, May 7, 2010

Vector fields and nonlinear systems

1. Autonomous systems
2. Vector fields
3. Equilibria
4. Linearization

[1] Introduction to general nonlinear autonomous systems.

Recall that an ODE is autonomous if x' depends only on x and not on t :

$$x' = f(x)$$

For example, I know an island in the St Lawrence River in upstate New York where there are a lot of deer [Slide]. When there aren't many of them, they multiply with growth rate k ; $x' = kx$. Soon, though, they push up against the limitations of the island; the growth rate is a function of the population, and we might take it to be $k(1-(x/a))$ where a is the maximal sustainable population of deer on the island. So the equation is

$$x' = k(1-(x/a))x, \text{ the "logistic equation."}$$

On this particular island, $k = 3$ and $a = 3$, so $x' = (3-x)x$. (Maybe the units are hundreds of deer, and years).

There are "critical points" at $x = 0$ and $x = 3$.

When $0 < x < 3$, $x' > 0$.

When $x > 3$, $x' < 0$, and, unrealistically, when $x < 0$, $x' < 0$ too.

I drew some solutions, and then recalled the phase line:

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One day, a wolf [Slide] swims across from the neighboring island, pulls himself up the steep rocky coast, shakes the water off his fur, and sniffs the air. Two wolves, actually.

Wolves eat deer, and this has a depressing effect on the growth rate of deer. Let's model it by

$$x' = (3-x-y)x$$

where y measures the population of wolves.

Now, wolves in isolation follow a logistic equation too, say

$$y' = (1-y)y \quad (\text{no deer})$$

But the presence of deer increases their growth rate, say

$$y' = (1-y+x)y$$

We have a *nonlinear autonomous system*

$$\begin{aligned}x' &= (3-x-y)x \\y' &= (1-y+x)y\end{aligned}$$

[2] Vector fields. The general case would be

$$\begin{aligned}x' &= f(x,y) \\y' &= g(x,y)\end{aligned}$$

We can visualize this system in the following way. The pair of populations determines a point on the plane, $[x; y]$. The population evolves through time, tracing a curve on the plane: $u(t) = [x(t); y(t)]$. What we know about this curve is that its velocity vector,

$$u'(t) = [x'(t); y'(t)]$$

is a function the current position on the plane:

$$u'(t) = F(u(t)), \quad F(x,y) = [f(x,y); g(x,y)]$$

This data associates a vector to each point in the plane. It is a "vector field," as studied in 18.02.

An autonomous system is the same thing as a vector field. A solution is a vector valued function $u(t)$ whose velocity vector at each point it goes through is given by the value of the vector field at that point.

We have not been drawing the vector fields of the linear systems we've been studying, but they are there. For example $A = [0 \ -1; 1 \ 0]$ determines the vector field with value at $[x;y]$ given by

$$A[x;y] = [-y;x]$$

This is the vector with the same length as $[x;y]$ but which points at right angles to the radius vector. $\text{Tr}(A) = 0$, $\text{Det}(A) = 1$, so we have a center. In fact the trajectories are circles.

I displayed the vector field associated to the deer/wolf system, using the web version of pplane, <http://math.rice.edu/~dfield/dfpp.html>.

The trajectories are more complicated, and I clicked up a bunch of them. This is the "phase portrait" of the autonomous system.

Note that the phase line of the deer-only system occurs along the x-axis, and the wolf-only system along the y-axis. The "phase line" we studied back in section 1 is a 1-dimensional phase diagram.

[3] Now, again, we are interested in equilibria. There seem to be four of them in this example. The behavior near them looks pretty similar to the linear phase portraits we have just been studying: I see a node near $(0,0)$, saddles on the x- and y-axes, and a spiral out in the middle somewhere.

This is a deep and important fact: usually, near an equilibrium an autonomous system is well approximated by a linear system, which we can

then analyze using eigenvectors and eigenvalues. You can see the eigenvectors in the nonlinear phase diagram!

Let's find out where the equilibria, or critical points, are located.

First think about where the horizontal component of the vector field vanishes:

$$x' = 0, \text{ that is } (3-x-y)x = 0.$$

This happens when either $x = 0$ or $3-x-y = 0$; that is, the vector field is vertical along those two lines.

The vertical component vanishes where

$$y' = 0, \text{ that is } (1-y+x)y = 0.$$

This happens when either $y = 0$ or $1-y+x = 0$; that is, the vector field is horizontal along those two lines.

The vector field is both vertical and horizontal exactly when it vanishes. There are four places where that happens:

$$(0,0), (3,0), (0,1),$$

and the place where both $y = 3-x$ and $y = 1+x$, which is at $(1,2)$.

[4] Linearization. Let's see what happens near $(0,0)$, for a start:

Expanding out,

$$\begin{aligned} x' &= f(x,y) = 3x - x^2 - xy \\ y' &= g(x,y) = y + xy - y^2 \end{aligned}$$

Near the origin the quadratic terms in the vector field are insignificant, so "to first order"

$$\begin{aligned} x' &= 3x \\ y' &= y \end{aligned}$$

The deer and wolf populations both expand by natural growth. They are "decoupled." This is a homogeneous linear system with matrix $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$. The eigenvalues are 1 and 3, so we have a node. The eigenline for the smaller eigenvalue is the y axis, so this is the line to which all solutions (except for the other normal mode!) all solutions become tangent as they approach the origin.

At some other point $P = (a,b)$, we have the "tangent line (or plane) approximation:

$$f(x,y) \sim f(a,b) + \left(\frac{\partial f}{\partial x}\right)\bigg|_P (x-a) + \left(\frac{\partial f}{\partial y}\right)\bigg|_P (y-b)$$

$$g(x,y) \sim g(a,b) + \left(\frac{\partial g}{\partial x}\right)\bigg|_P (x-a) + \left(\frac{\partial g}{\partial y}\right)\bigg|_P (y-b)$$

Suppose (a,b) is a critical point for, so the $f(a,b) = g(a,b) = 0$.
Use a new coordinate system:

$$u = [x-a ; y-b]$$

Then

$$u' \sim J(a,b) u$$

where J is the Jacobian matrix,

$$J(x,y) =$$

$$\begin{array}{cc} \text{partial } f/\text{partial } x & \text{partial } f/\text{partial } y \\ \text{partial } g/\text{partial } x & \text{partial } g/\text{partial } y \end{array}$$

$$\text{For us: } J = [3-2x-y , -x ; y , 1+x-2y]$$

$$\text{So } A = J(1,2) = [3-2-2 , -1 ; 2 , 1+1-4] = [-1 , -1 ; 2 , -2] .$$

The point is that near to $(1,2)$ the system behaves very much like the linear system with this matrix of coefficients.

The characteristic polynomial is

$$p_A(\lambda) = \lambda^2 + 3\lambda + 4 = (\lambda + (3/2))^2 + (7/4)$$

so the eigenvalues are $-(3/2) \pm i \sqrt{7}/2$.

We have a stable spiral, rotating counterclockwise since the lower left entry is positive. I sketched this and observed that it does correspond to the nonlinear phase portrait near that critical point.

We have learned that the populations of deer and of wolves converge to these stable levels no matter what the initial populations are (as long as they are positive). Also, as the population levels approach this limiting value, they oscillate about them, decaying exponentially like $e^{-3t/2}$ and with period

$$(2\pi) / (\sqrt{7}/2) \sim 4.75.$$

The entire upper right quadrant is a "basin," with the equilibrium $(1,2)$ a "sink" or "attractor."

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