1. Notation and language

1.1. **Numbers.** We'll write $\mathbb{R}$ for the set of all real numbers. We often think of the set of real numbers as the set of points on the number line.

An **interval** is a subset I of the number line such that if $a, c \in I$, and b is a real number between a and c, then $b \in I$. There are several types of intervals, each with a special notation:

$(a, c) = \{b \in \mathbb{R} : a < b < c\}$, open intervals;

$(a, c] = \{b \in \mathbb{R} : a < b \leq c\}$ and

$[a, c) = \{b \in \mathbb{R} : a \leq b < c\}$, half-open intervals; and

$[a, c] = \{b \in \mathbb{R} : a \leq b \leq c\}$, closed intervals.

There are also unbounded intervals,

$(a, \infty) = \{b \in \mathbb{R} : a < b\}$,

$[a, \infty) = \{b \in \mathbb{R} : a \leq b\}$,

$(-\infty, c) = \{b \in \mathbb{R} : b < c\}$,

$(-\infty, c] = \{b \in \mathbb{R} : b \leq c\}$, and

$(-\infty, \infty) = \mathbb{R}$, the whole real line. The symbols ∞ and $-\infty$ do *not* represent real numbers. They are merely symbols so that $-\infty < a$ is a true statement for every real number a, as is $a < \infty$.

1.2. **Dependent and independent variables.** Most of what we do will involve *ordinary* differential equations. This means that we will have only *one* independent variable. We may have several quantities depending upon that one variable, and we may wish to represent them together as a vector-valued function.

Differential equations arise from many sources, and the independent variable can signify many different things. Nonetheless, very often it represents time, and the dependent variable is some dynamical quantity which depends upon time. For this reason, in these notes we will pretty systematically use t for the independent variable, and x for the dependent variable.

Often we will write simply x, to denote the entire function. The symbols x and $x(t)$ are synonymous, when t is regarded as a variable.

We generally denote the derivative with respect to t by a dot:

$$\dot{x} = \frac{dx}{dt},$$

and reserve the prime for differentiation with respect to a spatial variable. Similarly,

$$\ddot{x} = \frac{d^2x}{dt^2}\,.$$

1.3. **Equations and Parametrizations.** In analytic geometry one learns how to pass back and forth between a description of a set by means of an *equation* and by means of a *parametrization*.

For example, the *unit circle*, that is, the circle with radius 1 and center at the origin, is defined by the equation

$$x^2 + y^2 = 1\,.$$

A *solution* of this equation is a value of (x, y) which satisfies the equation; the set of solutions of this equation is the unit circle. Any set will be the solution set of many different equations; for example, this same circle is also the set of points (x, y) in the plane for which $x^4 + 2x^2y^2 + y^4 = 1$.

This solution set is the same as the set parametrized by

$$x = \cos\theta\,, \quad y = \sin\theta\,, \quad 0 \le \theta < 2\pi\,.$$

The set of *solutions* of the equation is the set of *values* of the parametrization. The angle θ is the *parameter* which specifies a solution.

An **equation** is a *criterion*, by which one can decide whether a point lies in the set or not. $(2, 0)$ does not lie on the circle, because it doesn't satisfy the equation, but $(1, 0)$ does, because it does satisfy the equation.

A **parametrization** is an *enumeration*, a listing, of all the elements of the set. Usually we try to list every element only once. Sometimes we only succeed in picking out some of the elements of the set; for example

$$y = \sqrt{1 - x^2}\,, \quad -1 \le x \le 1$$

picks out the upper semicircle. For emphasis we may say that some enumeration gives a *complete parametrization* if every element of the set in question is named; for example

$$y = \sqrt{1 - x^2}\,, \quad -1 \le x \le 1\,, \quad \text{or} \quad y = -\sqrt{1 - x^2}\,, \quad -1 < x < 1\,,$$

is a complete parametrization of the unit circle, different from the one given above in terms of cosine and sine.

Usually the process of "solving" and equation amounts to finding a parametrization for the set defined by the equation. You could call a

parametrization of the solution set of an equation the "general solution" of the equation. This is the language used in Differential Equations.

1.4. **Parametrizing the set of solutions of a differential equation.** A *differential* equation is a stated relationship between a function and its derivatives. A *solution* is a function satisfying this relationship. (We'll emend this slightly at the end of this section.)

For a very simple example, consider the differential equation

$$\ddot{x} = 0\,.$$

A *solution* is a function which satisfies the equation. It's easy to write down many such functions: any function whose graph is a straight line satisfies this ODE.

We can enumerate all such functions: they are

$$x(t) = mt + b$$

for m and b arbitrary real constants. This expression gives a *parametrization* of the set of solutions of $\ddot{x} = 0$. The constants m and b are the *parameters*. In our parametrization of the circle we could choose θ arbitrarily, and analogously now we can choose m and b arbitrarily; for any choice, the function $mt + b$ is a solution.

Warning: If we fix m and b, say $m = 1$, $b = 2$, we have a specific line in the (t, x) plane, with equation $x = t + 2$. One can parametrize this line easily enough; for example t itself serves as a parameter, so the points $(t, t+2)$ run through the points on the line as t runs over all real numbers. This is an *entirely different* issue from the parametrization of solutions of $\ddot{x} = 0$. Be sure you understand this point.

1.5. **Solutions of ODEs.** The basic existence and uniqueness theorem for ODEs is the following. Suppose that $f(t, x)$ is continuous in the vicinity of a point (a, b). Then there exists a solution to $\dot{x} = f(t, x)$ defined in some open interval containing a, and it's unique provided $\partial f/\partial x$ exists.

There are certainly subtleties here. But some things are obvious. The "uniqueness" part of this theorem says that knowing $x(a)$ for one value $t = a$ is supposed to pick out a single solution: there's supposed to be only one solution with a given "initial value." Well, look at the ODE $\dot{x} = 1/t$. The solutions can be found by simply integrating: $x = \ln|t| + c$. This formula makes it look as though *the* solution with $x(1) = 0$ is $x = \ln|t|$. But in fact there is no reason to prefer this

to the following function, which is also a solution to this initial value problem, for any value of c:

$$x(t) = \begin{cases} \ln t & \text{for } t > 0\,, \\ \ln(-t) + c & \text{for } t < 0\,. \end{cases}$$

The gap at $t = 0$ means that the values of $x(t)$ for $t > 0$ have no power over the values for $t < 0$.

For this reason it's best to declare that a *solution* to an ODE *must be defined on an entire interval*. The graph has to be a connected curve.

Thus it is more proper to say that the solutions to $\dot{x} = 1/t$ are $\ln(t)+c$ for $t > 0$ and $\ln(-t) + c$ for $t < 0$. The *single formula* $\ln|t| + c$ actually describes *two solutions* for each value of c, one defined for $t > 0$ and the other for $t < 0$. The solution with $x(1) = 0$ is $x(t) = \ln t$, with domain of definition the interval $(0, \infty)$.

18.03 Differential Equations

Spring 2010