# Recitation 9, March 4, 2010

## Exponential and sinusoidal input signals.

**1.** Find $A$ so that $A\sin(3t)$ is a solution of $\ddot{x} + 4x = \sin(3t)$. What is the general solution?

We can find this by brute force. If $x = A\sin(3t)$, then $\ddot{x} = -9A\sin(3t)$, so $\ddot{x} + 4x = -5A\sin(3t)$. Therefore, when $A = -1/5$, $x_p(t) = -\sin(3t)/5$ is a solution of the given equation.

To find the general solution, we add to $x_p$ the general solution to the homogeneous equation $\ddot{x} + 4x = 0$. The characteristic polynomial is $p(s) = s^2 + 4$, with roots $\pm 2i$, so the general solution to $\ddot{x} + 4x = 0$ is $C_1\sin(2t) + C_2\cos(2t)$. Therefore, the general solution to $\ddot{x} + 4x = \sin(3t)$ is given by $-\sin(3t)/5 + C_1\sin(2t) + C_2\cos(2t)$.

**2.** For $\omega \geq 0$, find $A$ such that $A\cos(\omega t)$ is a solution of $\ddot{x} + 4x = \cos(\omega t)$. Graph the input signal $\cos(\omega t)$ and the solution $A\cos(\omega t)$ for $\omega = 0$, $\omega = 1$, and $\omega = 3$. Sketch a graph of $A$ as a function of $\omega$, as $\omega$ ranges from 0 to 5. Where does resonance occur? What is the significance of the sign of $A$?

If $x = A\cos(\omega t)$, then taking derivatives gives us $\ddot{x} = -\omega^2 A\cos(\omega t)$, and $\ddot{x} + 4x = (4 - \omega^2)A\cos(\omega t)$. Then $A = \frac{1}{4-\omega^2}$. When $\omega = 0$, $A = 1/4$, the input is constant 1 and the solution is constant $1/4$; when $\omega = 1$, $A = 1/3$, the input is $\cos(t)$ with the solution being$\cos(t)/3$; when $\omega = 3$, $A = -1/5$, the input is $\cos(3t)$ and the solution is $-\cos(3t)/5$. The graph of $A = A(\omega) = 1/(4 - \omega^2)$ in the range of $\omega \in [0, 5]$ has two branches: when $0 \leq \omega < 2$, $A(\omega)$ increases from $1/4$ to positive infinity; when $2 < \omega \leq 5$, $A(\omega)$ increases from negative infinity to $-\frac{1}{21}$. Resonance occurs at the pole, when $\omega = 2$. $A > 0$ means in phase, $A < 0$ means antiphase, i.e., 180 degrees out of phase.

**3.** Find an exponential solution of $\dfrac{d^4x}{dt^4} - x = e^{-2t}$.

The characteristic polynomial of the homogeneous equation is given by $p(s) = s^4 - 1$. Since $p(-2) = 15 \neq 0$, the exponential response formula gives the solution $\frac{e^{-2t}}{p(-2)} = \frac{e^{-2t}}{15}$.

**4.** Find a sinusoidal solution of $\dfrac{d^4x}{dt^4} - x = \cos(2t)$.

We choose an exponential input function whose real part is $\cos(2t)$, namely $e^{2it}$. Since $p(s) = s^4 - 1$ and $p(2i) = 15 \neq 0$, the exponential response formula yields the solution $\frac{e^{2it}}{15}$. A sinusoidal solution to the original equation is given by the real part: $\frac{\cos(2t)}{15}$.

**5.** Find the general solution of the differential equations in **(3)** and **(4).**

To get the general solution, we take the sum of the general solution to the homogeneous equation and the particular solution to the original equation. The homogeneous equation corresponding to both (3) and (4) is $\frac{d^4x}{dt^4} - x = 0$. The characteristic polynomial $p(s) = s^4 - 1$ has 4 roots: $\pm 1$, $\pm i$. So the general solution to $\frac{d^4x}{dt^4} - x = 0$ is given by $C_1e^t + C_2e^{-t} + C_3\cos(t) + C_4\sin(t)$ for arbitrary real constants $C_1, C_2, C_3, C_4$.

Therefore, the general solution to the equation in (3) is $\frac{e^{-2t}}{15} + C_1e^t + C_2e^{-t} + C_3\cos(t) + C_4\sin(t)$.

The solution to the equation in (4) is $\frac{\cos(2t)}{15} + C_1e^t + C_2e^{-t} + C_3\cos(t) + C_4\sin(t)$.

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