18.034 Honors Differential Equations
Spring 2009

UNIT I: FIRST-ORDER DIFFERENTIAL EQUATIONS

We set forth fundamental principles in the analysis of differential equations. We illustrate the use of integration to find the solutions of first-order linear differential equations and special classes of first-order nonlinear differential equations, called separable equations. Substitution techniques are used in studying linear fractional equations and special kind of second-order differential equations.

LECTURE 1. INTEGRATION AND SOLUTIONS

We recall *the fundamental theorem of calculus*

$$\frac{d}{dx}\int_{x_0}^{x} f(s)ds = f(x), \tag{1.1}$$

if f is continuous on an interval $x_0 \in I$. A solution of the differential equation

$$\frac{dy}{dx} = f(x) \tag{1.2}$$

is the function $y = \phi(x)$ which satisfies the differential equation on I. Upon inspection of (1.1), then, $y = \displaystyle\int_{x_0}^{x} f(s)ds$ is a solution of (1.2). This leads to an existence result.

Theorem 1.1. *If $f(x)$ is continuous on an internval $x_o \in I$ then given an arbitrary number y_0 there exists a unique solution of* (1.2) *satisfying $y(x_0) = y_0$. The solution is given as*

$$y(x) = y_0 + \int_{x_0}^{x} f(s)ds.$$

Exercise. Prove the uniqueness.

Remark. 1. The theorem specifies the interval of existence ($x_0 \in I$) and the class of functions considered (the class of continuous functions). It asserts the *existence* and *uniqueness* of a solution, prescribed the initial condition $y(x_0) = y_0$.

2. In the statement of the theorem, the interval of existence is I, regardless of the initial condition. It is a special property of linear equations. For nonlinear equations, in general, the interval of existence depends on the initial value, e.g. the solution of the initial value problem

$$\frac{dy}{dx} = y^2, \qquad y(0) = y_0 \neq 0,$$

is given as $y(x) = \dfrac{1}{(1/y_0) - x}$. It is defined on $x \in [0, 1/y_0)$ for $y_0 > 0$.

3. The definite integral $\displaystyle\int_{x_0}^{x} f(s)ds$ is defined as a limit of Riemann sums, as long as f is continuous; it doesn't need to find a formal expression for the indefinite integral $\int f(s)ds$ to give meaning to the definite integral, e.g. the *error function* $\text{erf}(x) = \int_0^x e^{-s^2} ds$ and the *sine integral* function $\text{Si}(x) = \int_0^x (\sin s)/s\, ds$ are commonly defined as definite integrals.

As an illustration, the solution of the initial value problem

$$\frac{dy}{dx} = \sin x^2, \qquad y(0) = 0$$

is given by the *Fresnel sine integral* function $\mathrm{S}(x) = \int_0^x \sin s^2\, ds$. There is no elementary function F such that $F'(x) = \sin x^2$, but the function $\mathrm{S}(x)$ defined as a definite integral gives a perfectly good function.

The preceding discussion leads to how to solve differential equations of the form (1.2) by inspection. For any x_0, one solution is the function $\displaystyle\int_{x_0}^{x} f(s)ds$. Other solutions are, then, obtained by adding an arbitrary constant C to this "particular" solution. Thus, the solutions of $y' = e^{-x^2}$ are the functions $y = \int e^{-s^2} ds = (\sqrt{\pi}/2)\mathrm{erf}(x) + C$. From any one solution curve of (1.2), the others are obtained by the vertical translations $(x, y) \mapsto (x, y + C)$ and they form a one-parameter family of curves, one for each value of the parameter C.

Quadrature. When the solution of a differential equation is expressed by a formula involving one or more integrations, it is said that the equation is solvable *by quadrature*. The term "quadrature" has its historical origin in the connection of integration with area. In plane geometry, a problem of quadrature, such as *quadrature of the circle* is a problem about the area of a plane figure. Not all differential equations can be solved by quadrature. In the following lecture, we will show that the first-order linear equation

$$y' + p(x)y = q(x)$$

can be solved by quadrature. But, the second-order differential equation

$$y'' + p(x)y' + q(x)y = r(x)$$

cannot be solved, in general, by quadrature, except for some special cases.

The next simples type of differential equation is

$$\frac{dy}{dx} = g(y). \tag{1.3}$$

Such a differential equation is invariant under horizontal translations $(x, y) \mapsto (x + c, y)$. Geometrically, it means that any horizontal line is cut by all solution curves at the same angle (such lines are called "isoclines"). Therefore, if $y = \phi(x)$ is a solution of (1.3), then so is $y = \phi(x + c)$ for any c. The differential equation (1.3) can be solved by writing it as $dy/g(y) = dx$ and integrating.

Example 1.2. Consider

$$\frac{dy}{dx} = y^2 - 1. \tag{1.4}$$

Since $y^2 - 1 = (y - 1)(y + 1)$, the constant functions $y = \pm 1$ are particular solutions of (1.4). They are called *steady states*, *stationary solutions* or *equilibria*, in the sense that these solutions are independent of x.

Next, if $|y| < 1$ then $y^2 - 1 < 0$, and follows $dy/dx < 0$. That is, the solution curve is decreasing. On the other hand, if $|y| > 1$, then $dy/dx = y^2 - 1 > 0$, and the solution curve is increasing. It gives us the qualitative behavior of solutions curves of (1.4).

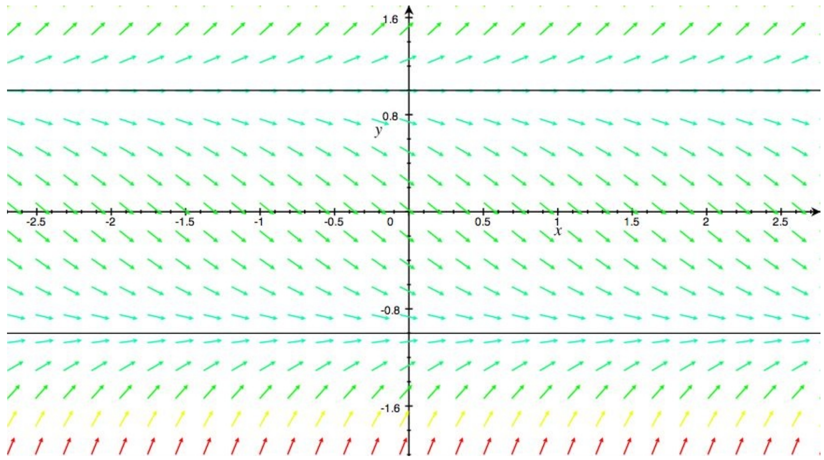

Figure 1.1. Qualitative behavior of solutions of $y' = y^2 - 1$.

Using the partial fractions and separating variables (we will discuss this technique in detail later), (1.4) is written as

$$2dx = dy\left(\frac{1}{y-1} - \frac{1}{y+1}\right).$$

Then, by integration, we obtain

$$y(x) = \frac{1 \pm e^{2(x-c)}}{1 \mp e^{2(x-c)}} = \left\{ \begin{array}{c} \tanh \\ \coth \end{array} \right\} (c-x).$$

This procedure of separating variables "loses" the particular solutions $y = \pm 1$, but it gives all other solutions.

Note that if $y = \phi(x)$ is a solution of (1.4) then so is $1/y = 1/\phi(x)$.

Exercise. Discuss $y' = y^3 - y$.