Solution to the Constant Coefficient First Order Equation

In this section we will consider the constant coefficient equation

$$\dot{y} + ky = q(t). \tag{1}$$

(Constant coefficient means k is a constant.)

Solving is easy using the integrating factor $u(t) = e^{\int k\,dt} = e^{kt}$ from session 5. We get the solution

$$\begin{aligned} y &= e^{-kt}\left(\int e^{kt}q(t)\,dt + c\right) &(2)\\ &= e^{-kt}\int e^{kt}q(t)\,dt + ce^{-kt}. &(3)\end{aligned}$$

As usual, we have a *particular* solution and a *homogeneous* solution, respectively

$$y_p(t) = e^{-kt}\int e^{kt}q(t)\,dt \quad \text{and} \quad y_h(t) = e^{-kt}.$$

The general solution to (1) is then

$$y(t) = y_p(t) + c\,y_h(t).$$

The Case k > 0.
If $k > 0$ the system in (1) models *exponential decay*. That is, when the input is 0 the system response is $y(t) = ce^{-kt}$, which *decays exponentially* to 0 as t goes to ∞.

In the general solution we call ce^{-kt} the **transient** because it goes to 0. The term $e^{-kt}\int e^{kt}q(t)\,dt$ is called the **steady-state** or **long-term** solution. That is, cy_h is the transient and y_p is the steady-state solution.

The value of c in (2) is determined by the initial value $y(0)$. The initial condition only affects the transient and not the long-term behavior of the solution. No matter what the initial condition, every solution goes asymptotically to the steady-state. That is, all solution curves approach the steady-state as $t \to \infty$.

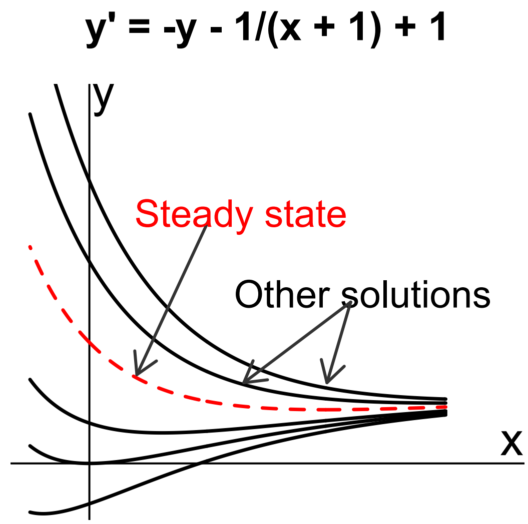

Fig. 1. In the case $k > 0$ all solutions go asymptotically to the steady-state.

Since all the solutions approach each other, there is no precise way to choose the one we call the steady-state. In fact, we can *choose any one* to be the steady-state solution. Generally, we just choose the simplest looking solution.

The case $k \leq 0$.
When $k \leq 0$ the homogeneous solution e^{-kt} does not go asymptotically to 0. In other words it is not transient. In this case it does not make sense to talk about the steady-state solution.