18.03SC Practice Problems 2

Direction fields, integral curves, isoclines, separatrices, funnels

This session is accompanied by the `Isoclines` Mathlet in the Mathlet Gallery

A *direction field* of a differential equation $\frac{dy}{dx} = F(x, y)$ is a visual representation of the differential equation in the plane by arrows with direction (signed slope) given by the value of F at their base point. A direction field is also called a slope field. This is the terminology used in the `Isoclines` Mathlet.

An $(m-)$*isocline* of the differential equation $\frac{dy}{dx} = F(x, y)$ is the solution set of the equation $F(x, y) = m$, for some fixed m. The 0-isocline, which also called the *null-cline,* is especially important because it is the set of all the constant solutions of the differential equation. A good way to create direction fields is to plot a few isoclines, making sure to include the nullcline.

An *integral curve* is the graph of a solution to the equation. At every point on an integral curve, the slope of the tangent line to the curve is given by the value of F at that point.

As an example, take the ODE

$$\frac{dy}{dx} = x - 2y.$$

1. Draw a big axis system and plot some isoclines, especially the nullcline. Use them to illustrate the direction field. Using the direction field, plot a few solutions. Try to do this by hand first. Later you might want to refer to the `Isoclines` Mathlet.

2. One of the integral curves seems to be a straight line. Is this true? What straight line is it? (i.e. for what m and b is $y = mx + b$ a solution?)

3. In general – for the general differential equation $\frac{dy}{dx} = F(x, y)$ – if a straight line is an integral curve, how is it related to the isoclines of the equation? What happens in our example?

4. It seems that all the solutions become asymptotic to each other as $x \to \infty$. We will see later that this is true, but for now explain why solutions get trapped between parallel lines of some fixed slope.

5. Where are the critical points of the solutions of $y' = x - 2y$? How many critical points can a single solution have? For what values of y_0 does the solution y with $y(0) = y_0$ have a critical point? When there is one, is it a minimum or a maximum? You can see an answer to this from your picture. Can you also use the second derivative test to be sure?

6. For another example, take $\frac{dy}{dx} = y^2 - x^2$. (This is also on the `Isoclines` Mathlet.) Again, make a big picture of some isoclines and use them to sketch the direction field, and then sketch a few solutions.

7. A "separatrix" is a solution such that solutions above it have a fate (as x increases) entirely different from solutions below it. The equation $\frac{dy}{dx} = y^2 - x^2$ exhibits a separatrix. Sketch it and describe the differing behaviors of solutions above it and below it.

8. The equation $y' = y^2 - x^2$ also exhibits a "funnel," where solutions get trapped as x increases, and many solutions are asymptotic to each other. Explain this using a couple of isoclines. There is a function with a simple formula (not a solution to the equation, though) which all these trapped solutions get near to as x gets large. What is it?