PROFESSOR: The isoclines applet. Let's explore graphs of solutions of differential equations using direction fields and isoclines.

The first thing to do is to choose an equation from the pull-down menu here. And for this demonstration, I will choose the differential equation y prime equals y squared minus x.

In the window at left, you can see the direction or slope field drawn. This is a representation of the differential equation. When I move the cursor over this window, a readout of the x- and y-coordinates shows up on the right-hand side of the screen. This is very useful in making measurements.

If I click on the graphing window, the solution through the point that I've clicked on is drawn. This is an interesting function, probably one you've never seen before. Certainly, it's not an elementary function.

Clicking some more points makes more solutions appear. In fact, every point on this plane has exactly one solution through it. This is the meaning of the existence and uniqueness theorem for differential equations.

Now, let's clear the solutions using this button down here, Clear Solutions button. You see a blank screen again.

You can control how the slope field looks by using this Slope Field toggle here. You can make them blank, or bright for a display, for example. Or the way it was originally. And I think I'm going to turn it off altogether, so now we're faced with a blank screen.

Now, how would you go about drawing some solutions to this differential equation by hand just knowing the differential equation and this blank screen? Well, isoclines give you a good way of doing this, and they reveal things about the qualitative behavior of the differential equation as well.

An isocline is the subset of the plane where the slope field takes on a given value. I can choose that value, m, using this slider here. And when I click here and move this, the isocline with the given value of m is drawn in yellow on the screen.

And you can see the value of the slope field drawn as well. So here is the value is 1, and the slope field is given by little intervals of value 1, of slope 1.

I can choose that value using the slider marked m. If I click on the handle and drag it, you see the isocline for the corresponding value of m drawn on the screen.

And on the isocline is also drawn the direction fields. So when m is 2, the direction field has slope 2. And when I drag the slider back down to 0, the isocline value 0 has a horizontal direction field marked along it.

Each one of these is a curve where y squared minus x equals m, or x equals y squared minus m. This is a parabola lying on its side with the vertex at x equals negative m.

Let's draw the isocline for value 1, for example. I've now released the mouse key, and the isocline is left behind. And I can easily draw in some other isoclines as well by clicking on different values of m and releasing the mouse key.

So once you've drawn several of these isoclines on the plane, it's pretty easy to envision what the solutions will look like to this curve. You just have to thread your way along the part of the direction field that you've drawn. I can check that by clicking on the screen and drawing a solution in.

This is quite easy to do by hand. It's easy to draw the isoclines and then sketch a solution accordingly.

But you can see other things as well from isoclines. I'm going to clear all of these and redraw the m equals 0 isocline on the screen. And I think I'm going to redraw the slope field as well.

Now, critical points of a function occur where the derivative is equal to 0. And the derivative is exactly what we know about the solution to a differential equation. So the critical points, the minima or maxima of solutions to this differential equation, occur when the solution crosses the m equals 0 isocline, also known as the nullcline.

All maxima and minima of solutions to this differential equation occur along this particular yellow parabola. I can check that by drawing in some solutions, and you can see that these functions have maxima which occur just along that parabola. If the solutions miss the parabola, they don't have any critical points.

Another thing you can see from this picture is that apparently many of the solutions to this differential equation cluster near to this branch of the nullcline. And maybe we can see why this is using the isocline picture.

I'm going to clear the solutions now to make the picture clearer and draw in one more isocline, namely the m equals minus 1 isocline.

Now, suppose that a solution finds itself below the m equals minus 1 isocline. So it's in here somewhere. Can it ever cross the m equals minus 1 isocline? Once it's below it, can it ever cross it?

Well, it's below it. So if it crosses the m equals minus 1 isocline, it must cross it with a slope which is bigger than the slope of that yellow curve. But when it crosses it, it also has to cross it with slope minus 1, because this is the m equals minus 1 isocline. But the slope of this yellow parabola is bigger than minus 1 along here, and so the solution curve can never cross it.

Similarly, suppose that you have a solution which is above the nullcline. Can it ever cross the nullcline? Well, if it crosses the nullcline, it must cross it from above, so when it crosses it, its slope must be less than the slope of the nullcline, which is negative as you can see. But when the solution crosses the nullcline, it must cross it with slope 0. And so, that can't happen.

And so you see, if a solution is between those two isoclines, then it stays between them forever more. It's trapped between them. This is called a funnel. It's trapped between these two things, and gets closer and closer, because these two isoclines become asymptotic as x gets large.

These are called fences as well. Once a solution is in here, it can't cross either of these two fences.

This let's us estimate the value of solutions for large x. These solutions, anyway, for large x. For example, if x is equal to 100, the solution has to be bigger than the value of this parabola, which is minus 10, and less than the value along this parabola, which is minus the square root of 99. So you get a very good estimate for the value of solutions for large x from these kinds of considerations.

One more thing you can see from this applet is this. If a solution is well above the nullcline, it gets caught in this powerful updraft and goes off to infinity. In fact, all these solutions become tangent to vertical lines. They don't continue for all large x. They blow up in finite time.

On the other hand, if you're just a little bit smaller than these solutions, then you get solutions which cross the nullcline, get trapped into this parabolic region and fall down in between our funnel, down here, and so become asymptotic to minus the square root of x when x gets large.

There's just one solution which doesn't do either of these very different behaviors, and it's right along here. It becomes asymptotic to the positive branch of the parabola, and every solution above it blows up in finite time, and every solution below it falls down and becomes asymptotic to minus the square root of x.

This special solution-- which doesn't do either of those two behaviors, but continues to exist for all positive values of x and become asymptotic to the square root of x-- this is called the separatrix. It separates solutions showing two very different types of behavior, the ones that fall down and the ones that blow up.

Well, these are just a few of the things you can understand using this applet. Play with different menu items. Open a copy of this applet in your browser window. Play with some other menu items. Maybe the default item at the bottom of this pull down menu down here.

Are there funnels? Are there separatrices? What happens to solutions as x goes to minus infinity rather than x equals plus infinity? Where are the critical points of solutions of this equation?