18.03SC Practice Problems 28

Inverse Laplace transform

Solution suggestions

1. *Find the inverse Laplace transform for each of the following.*

$$\frac{2s+1}{s^2+9} \quad , \qquad \frac{s^2+2}{s^3-s} \quad , \qquad \frac{2}{s^2(s-1)}.$$

For each of these expressions, we will first use the method of partial fractions to break up the expression, and then use formula tables and linearity to read off its inverse transform.

The first expression breaks up as

$$\frac{2s+1}{s^2+9} = 2\frac{s}{s^2+9} + (1/3)\frac{3}{s^2+9},$$

so it has inverse transform

$$\mathcal{L}^{-1}\left[\frac{2s+1}{s^2+9}\right] = 2\cos(3t) + \frac{1}{3}\sin(3t),$$

for $t > 0$.

The denominator in the second expression factors as $s^3 - s = s(s-1)(s+1)$. So it has a partial fractions decomposition of the form

$$\frac{s^2+2}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} \quad .$$

We use the Heaviside cover up method to determine that the coefficients here are

$$A = \frac{0^2+2}{(0-1)(0+1)} = -2,$$

$$B = \frac{1^2+2}{(1)(1+1)} = \frac{3}{2},$$

and

$$C = \frac{(-1)^2+2}{(-1)(-1-1)} = \frac{3}{2},$$

so the inverse transform of the second expression is

$$\mathcal{L}^{-1}\left[\frac{s^2+2}{s^3-s}\right] = \mathcal{L}^{-1}\left[\frac{-2}{s} + \frac{3/2}{s-1} + \frac{3/2}{s+1}\right] = -2 + \frac{3}{2}e^t + \frac{3}{2}e^{-t},$$

for $t > 0$.

The denominator in the third expression is $s^2(s-1)$, which has a factor of multiplicity two. So this time the partial fraction decomposition has the slightly more complicated form

$$\frac{2}{s^2(s-1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s-1}.$$

We can uncover the leading coefficients A and C by Heaviside as before, and use them to solve for the remaining coefficient:

$$A = \frac{2}{0-1} = -2$$

and

$$C = \frac{2}{(1)^2} = 2,$$

so, from the original equation,

$$2 = -2(s-1) + Bs(s-1) + 2s^2,$$

and $B = -2$.

We can now read off that the inverse transform of the third expression is

$$\mathcal{L}^{-1}\left[\frac{2}{s^2(s-1)}\right] = \mathcal{L}^{-1}\left[\frac{-2}{s} + \frac{-2}{s^2} + \frac{2}{s-1}\right] = -2 - 2t + 2e^t,$$

for t>0.

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