## Pole Diagrams

### 1. Definition of the Pole Diagram

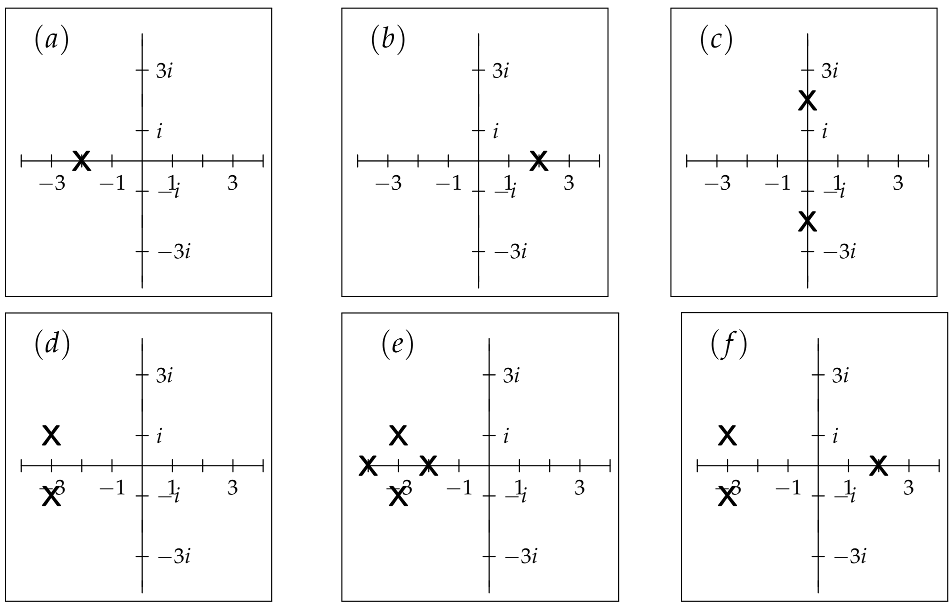
The **pole diagram** of a function $F(s)$ is simply the complex $s$-plane with an **X** marking the location of each pole of $F(s)$.

**Example 1.** Draw the pole diagrams for each of the following functions.

a) $F_1(s) = \frac{1}{s+2}$ b) $F_2(s) = \frac{1}{s-2}$ c) $F_3(s) = \frac{1}{s^2+4}$

d) $F_4(s) = \frac{s}{s^2+6s+10}$ e) $F_5(s) = \frac{1}{((s+3)^2+1)(s+2)(s+4)}$ f) $F_6(s) = \frac{1}{((s+3)^2+1)(s-2)}$

**Solution.**



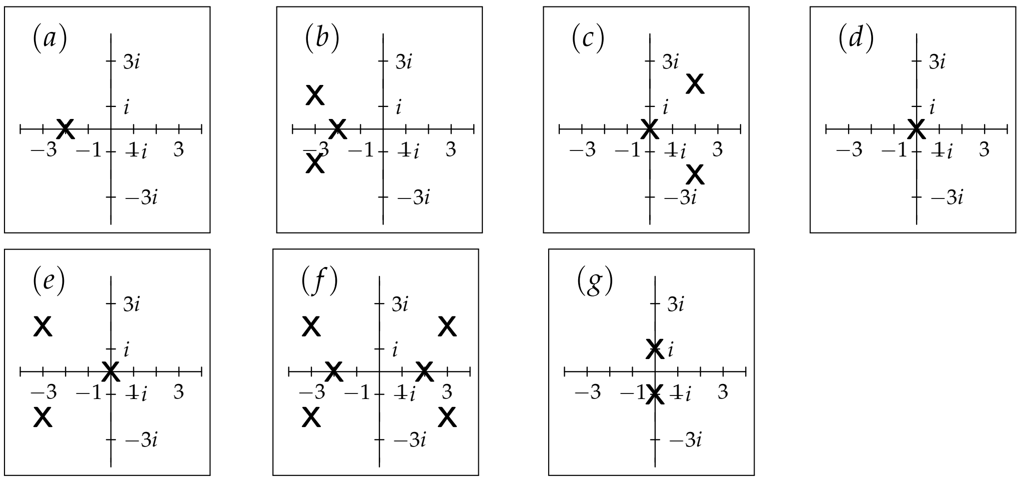

For (d) we found the poles by first completing the square: $s^2 + 6s + 10 = (s+3)^2 + 1$, so the poles are at $s = -3 \pm i$.

**Example 2.** Use the pole diagram to determine the exponential growth rate of the inverse Laplace transform of each of the functions in example 1.

**Solution.**
a) The largest pole is at -2, so the exponential growth rate is -2.
b) The largest pole is at 2, so the exponential growth rate is 2.
c) The poles are $\pm 2i$, so the largest real part of a pole is 0. The exponential growth rate is 0.
d) The largest real part of a pole is -3. The exponential growth rate is -3.
e) The largest real part of a pole is -2. The exponential growth rate is -2.

f) The largest real part of a pole is 2. The exponential growth rate is 2.

**Example 3.** Each of the pole diagrams below is for a function $F(s)$ which is the Laplace transform of a function $f(t)$. Say whether
(i) $f(t) \to 0$ as $t \to \infty$
(ii) $f(t) \to \infty$ as $t \to \infty$
(iii) You don't know the behavior of $f(t)$ as $t \to 0$,




**Solution.** a) Exponential growth rate is -2, so $f(t) \to 0$.
b) Exponential growth rate is -2, so $f(t) \to 0$.
c) Exponential growth rate is 2, so $f(t) \to \infty$.
d) Exponential growth rate is 0, so we can't tell how $f(t)$ behaves.
Two examples of this: (i) if $F(s) = 1/s$ then $f(t) = 1$, which stays bounded; (ii) if $F(s) = 1/s^2$ then $f(t) = t$, which does go to infinity, but more slowly than any positive exponential.
e) Exponential growth rate is 0, so don't know the behavior of $f(t)$.
f) Exponential growth rate is 3, so $f(t) \to \infty$.
g) Exponential growth rate is 0, so don't know the behavior of $f(t)$. (e.g. both $\cos t$ and $t \cos t$ have poles at $\pm i$.

## 2. The Pole Diagram for an LTI System

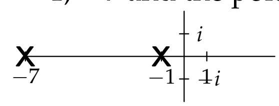
**Definition:** The pole diagram for an LTI system is defined to be the pole diagram of its transfer function.

**Example 4.** Give the pole diagram for the system

$$\ddot{x} + 8\dot{x} + 7x = f(t),$$

where we take $f(t)$ to be the input and $x(t)$ the output.

**Solution.** The transfer function for this system is $W(s) = \dfrac{1}{s^2+8s+1} = \dfrac{1}{(s+1)(s+7)}$. Therefore, the poles are $s = -1, -7$ and the pole diagram is




**Example 5.** Give the pole diagram for the system

$$\ddot{x} + 4\dot{x} + 6x = \dot{y},$$

where we consider $y(t)$ to be the input and $x(t)$ to be the output.

**Solution.** Assuming rest IC's, Laplace transforming this equation gives us $(s^2+4s+6)X = sY$. This implies $X(s) = \dfrac{s}{s^2+4s+6}Y(s)$ and the transfer function is $W(s) = \dfrac{s}{s^2+4s+6}$. This has poles at $s = -2 \pm \sqrt{2}\,i$.

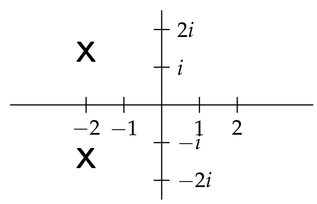



Figure: Pole diagram for the system in example 5.

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