Part I Problems and Solutions

Problems 1 and 2 are about the system

$$p(D)x = f(t) \tag{1}$$

with rest IC's and with input $f(t)$.

Problem 1: *In each of the following cases, find $p(D)$ such that $w(t)$ is the system unit impulse response.*

(a) $w(t) = e^{-at}$. **(b)** $w(t) = \frac{1}{3}\, e^{-t/2} \sin t$. **(c)** $w(t) = 1$.

Solution: a) $\mathcal{L}(w) = \dfrac{1}{p(s)} = \dfrac{1}{s+a} \;\Rightarrow\; \boxed{p(D) = D + a.}$

b) $\mathcal{L}(\sin t) = \dfrac{1}{s^2+1} \;\Rightarrow\; \mathcal{L}(w) = \dfrac{1}{3}\,\dfrac{1}{(s+1/2)^2+1} = \dfrac{1}{3s^2+3s+15/4} \;\Rightarrow$
$\boxed{p(D) = 3D^2 + 3D + 15/4.}$

c) $\mathcal{L}(w) = 1/s \;\Rightarrow\; \boxed{p(D) = D.}$

Problem 2: *For $p(D) = D^2 + 4$:*
(a) *Find the system function $W(s)$;*
(b) *Find the weight function $w(t)$;*
(c) *Write down the convolution integral formula for the solution to the IVP (1).*

Solution: We have
a) $p(s) = s^2 + 4 \;\Rightarrow\; W(s) = \dfrac{1}{p(s)} = \dfrac{1}{s^2+4}$.
b) $w(t) = \mathcal{L}^{-1}(W(s)) = \mathcal{L}^{-1}\left(1/(s^2+4)\right) = \dfrac{1}{2}\,\sin(2t)$.
c) $x_p(t) = w * f(t) = \dfrac{1}{2}\,\displaystyle\int_0^t \sin(2\tau)\, f(t-\tau)\, d\tau$.

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