Non-Existence of Limit Cycles

We turn our attention now to the negative side of the problem of showing limit cycles exist. Here are two theorems which can sometimes be used to show that a limit cycle does *not* exist.

1. Bendixson's Criterion

If f_x and g_y are continuous in a region R which is simply-connected (i.e., without holes), and

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \neq 0 \quad \text{at any point of } R,$$

then the system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \tag{1}$$

has no closed trajectories inside R.

Proof. Assume there is a closed trajectory C inside R. We shall derive a contradiction, by applying Green's theorem, in its normal (or flux) form. This theorem says

$$\oint_C (f\mathbf{i} + g\mathbf{j}) \cdot \mathbf{n}\, ds \quad \equiv \quad \oint_C f\, dy - g\, dx \quad = \quad \iint_D (\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y})\, dx\, dy\,. \tag{2}$$

where D is the region inside the simple closed curve C.

This however is a contradiction. Namely, by hypothesis, the integrand on the right-hand side is continuous and never 0 in R; thus it is either always positive or always negative, and the right-hand side of (2) is therefore either positive or negative.

On the other hand, the left-hand side must be zero. For since C is a closed trajectory, C is always tangent to the velocity field $f\mathbf{i} + g\mathbf{j}$ defined by the system. This means the normal vector $\mathbf{n}$ to C is always perpendicular to the velocity field $f\mathbf{i} + g\mathbf{j}$, so that the integrand $(f\mathbf{i} + g\mathbf{j}) \cdot \mathbf{n}$ on the left is identically zero.

This contradiction means that our assumption that R contained a closed trajectory of (1) was false, and Bendixson's Criterion is proved. □

Critical-point Criterion *A closed trajectory has a critical point in its interior.*

If we turn this statement around, we see that it is really a criterion for *non-existence*: it says that *if a region R is simply-connected (i.e., without holes) and has no critical points, then it cannot contain any limit cycles.* For if it did, the Critical-point Criterion says there would be a critical point inside the limit cycle, and this point would also lie in R since R has no holes.

(Note carefully the distinction between this theorem, which says that limit cycles enclose regions which *do* contain critical points, and the Poincare-Bendixson theorem, which seems to imply that limit cycles tend to lie in regions which *don't* contain critical points. The difference is that these latter regions always contain a hole; the critical points are in the hole. Example 1 illustrated this.

Example 2. For what a and d does $\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$ have closed trajectories?

Solution. By Bendixson's criterion, $a + d \neq 0 \quad \Rightarrow$ no closed trajectories.

What if $a + d = 0$? Bendixson's criterion says nothing. We go back to our analysis of the linear system. The characteristic equation of the system is

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0\,.$$

Assume $a + d = 0$. Then the characteristic roots have opposite sign if $ad - bc < 0$ and the system is a saddle; the roots are pure imaginary if $ad - bc > 0$ and the system is a center, which has closed trajectories. Thus

the system has closed trajectories $\quad \Leftrightarrow \quad a + d = 0, \quad ad - bc > 0.$