18.04 Practice problems exam 1, Spring 2018 Solutions

Problem 1. Complex arithmetic

(a) *Find the real and imaginary part of* $\dfrac{z+2}{z-1}$.

(b) *Solve* $z^4 - i = 0$.

(c) *Find all possible values of* $\sqrt{\sqrt{i}}$.

(d) *Express* $\cos(4x)$ *in terms of* $\cos(x)$ *and* $\sin(x)$.

(e) *When does equality hold in the triangle inequality* $|z_1 + z_2| \le |z_1| + |z_2|$?

(f) *Draw a picture illustrating the polar coordinates of z and $1/z$.*

Answers. (a) $\dfrac{x+2+iy}{x-1+iy} \cdot \dfrac{x-1-iy}{x-1-iy} = \dfrac{(x+2)(x-1)+y^2}{(x-1)^2+y^2} + i\,\dfrac{-3y}{(x-1)^2+y^2}$

(b) $i = e^{i(\pi/2+2n\pi)}$. So $z = i^{1/4} = e^{i(\pi/8+n\pi/2)} = \pm 1 e^{i\pi/8}, \pm i e^{i\pi/8}$.

(c) Same answer as part (b).

(d) Euler:

$$\cos(4x) + i\sin(4x) = e^{i4x} = (\cos(x) + i\sin(x))^4$$
$$= \cos^4(x) - 6\cos^2(x)\sin^2(x) + \sin^4(x) + i(4\cos^3(x)\sin(x) - 4\cos(x)\sin^3(x)).$$

Therefore, $\cos(4x) = \cos^4(x) - 6\cos^2(x)\sin^2(x) + \sin^4(x)$.

(e) When z_1 and z_2 have the same argument, i.e. are on the same ray from the origin.

(f)
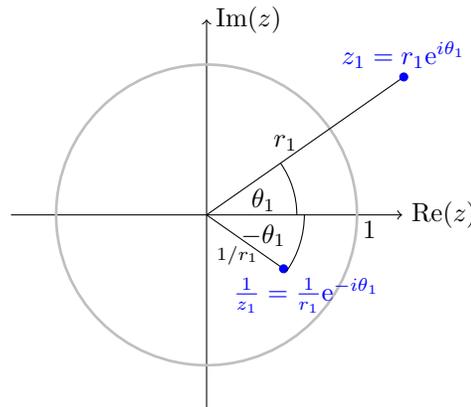
Problem 2. Functions

(a) *Show that* $\sinh(z) = -i\sin(iz)$.

Solution: $-i\sin(iz) = -i\dfrac{e^{i\cdot iz} - e^{-i\cdot iz}}{2i} = -\dfrac{e^{-z} - e^z}{2} = \sinh(z)$.   QED

(b) *Give the real and imaginary part of* $\cos(z)$ *in terms of x and y using regular and hyperbolic* $\sin$ *and* $\cos$.

Solution: We calculate this using exponentials.

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{-y+ix} + e^{y-ix}}{2}$$
$$= \frac{e^{-y}e^{ix} + e^{y}e^{-ix}}{2}$$
$$= \frac{e^{-y}(\cos(x) + i\sin(x)) + e^{y}(\cos(x) - i\sin(x))}{2}$$
$$= \frac{e^{-y} + e^{y}}{2}\cos(x) + i\frac{e^{-y} - e^{y}}{2}\sin(x)$$
$$= \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

Alternatively using the cosine addition formula:

$$\cos(z) = \cos(x + iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y).$$

(c) *Is it true that* $|a^b| = |a|^{|b|}$?

Solution: No: here's a counterexample: $|e^i| = 1$, but $|e|^{|i|} = e^1 = $ e.

Problem 3. Mappings
(a) *Show that the function* $f(z) = \frac{z-i}{z+i}$ *maps the upper half plane to the unit disk.*
(i) Show it maps the real axis to the unit circle.
(ii) Show it maps i to 0.
(iii) Conclude that the upper half plane is mapped to the unit disk.

Solution: (i) If z is real then $z - i = \overline{z + i}$, so numerator and denominator have the same norm, i.e. the fraction has norm 1. QED

(ii) Clearly $f(i) = 0$.

(iii) The boundary of the half plane is mapped to the boundary of the disk and a point in the interior of the half plane is mapped to the interior of the disk. This is enough to conclude that the image of the half plane is inside the disk.

Since it's easy to invert $u = f(z)$: $z = i\dfrac{1+u}{1-u}$. It is easy to see that the map is in fact one-to-one and onto the disk.

(b) *Show that the function* $f(z) = \dfrac{z+2}{z-1}$ *maps the unit circle to the line* $x = -1/2$.

Solution: We will learn good ways to manipulate expressions like this later in the course. Here we can do a direct calculation. Let $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$ be a point on the unit circle. Then

$$f(z) = \frac{\cos(\theta) + 2 + i\sin(\theta)}{\cos(\theta) - 1 + i\sin(\theta)} \cdot \frac{\cos(\theta) - 1 - i\sin(\theta)}{\cos(\theta) - 1 - i\sin(\theta)}$$
$$= \frac{(\cos(\theta) + 2)(\cos(\theta) - 1) + \sin^2(\theta) + i()}{(\cos(\theta) - 1)^2 + \sin^2(\theta)}$$
$$= \frac{-1 + \cos(\theta) + i(\cdots)}{2 - 2\cos(\theta)}$$
$$= -\frac{1}{2} + i\frac{(\cdots)}{2 - 2\cos(\theta)}$$

Here we left the imaginary part uncomputed because the question is to show that the real part is $-1/2$. Which we did!

Problem 4. Analytic functions

(a) *Show that $f(z) = e^z$ is analytic using the Cauchy Riemann equations.*

Solution: $e^z = e^x \cos(y) + i e^x \sin(y)$. Call the real and imaginary parts u and v respectively. Putting the partials in a matrix we have

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} e^x \cos(y) & -e^x \sin(y) \\ e^x \sin(y) & e^x \cos(y) \end{pmatrix}.$$

We see that $u_x = v_y$ and $u_y = -v_x$. Thus we have verified the Cauchy Riemann equations. So, $f(z)$ is analytic.

(b) *Show that $f(z) = \bar{z}$ is not analytic.*

Solution: $f(z) = x - iy = u + iv$, where $u = x$ and $v = -y$. Taking partials

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We see that $u_x \neq v_y$. So the Cauchy Riemann equations are not satisfied and so, $f(z)$ is not analytic.

(c) *Give a region in the z-plane for which $w = z^3$ is a one-to-one map onto the entire w-plane.*

Solution: Since z^3 triples arguments, we divide the plane into thirds and pick one third. We've chosen the shaded region in the figure below.
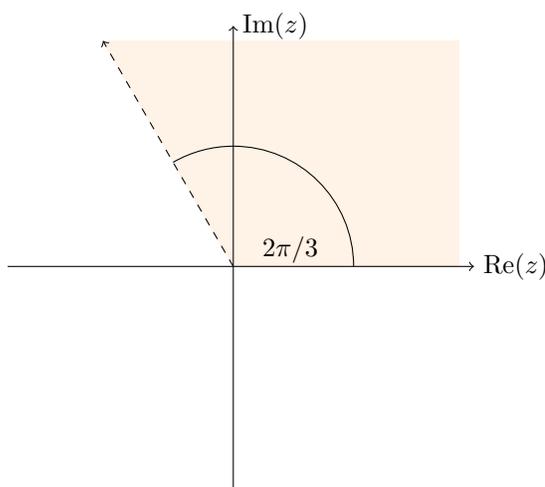
The region includes the the positive x-axis but not the dashed line.

(d) *Choose a branch of $z^{1/3}$ and a region of the z-plane where this branch is analytic. Do this so that the image under $z^{1/3}$ is contained in your region from part (c).*

Solution: We choose the branch of arg with $0 < \arg(z) < 2\pi$. So, the plane has a branch cut along the nonnegative real axis. Under $w = z^{1/3}$ the image points all have $0 < \arg(w) < 2\pi/3$, as required by the problem.
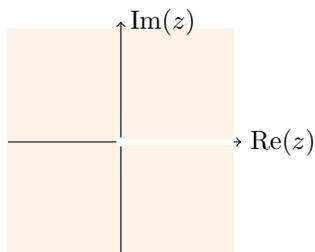
Problem 5. Line integrals

(a) *Compute* $\int_C x \, dz$, *where C is the unit square.*

Solution: First note that as a function x means $\mathrm{Re}(z)$. We do the integral for each of the four sides separately.
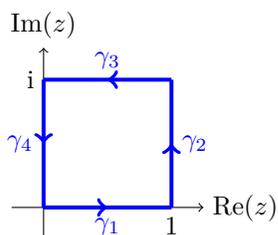
γ_1: $\gamma_1(t) = t$, with $0 \le t \le 1$. So, $\int_{\gamma_1} x \, dz = \int_0^1 t \, dt = 1/2$.

γ_2: $\gamma_2(t) = 1 + it$, with $0 \le t \le 1$. So, $\int_{\gamma_2} x \, dz = \int_0^1 1 \, i \, dt = i$.

γ_3: $\gamma_3(t) = 1 - t + i$, with $0 \le t \le 1$. So, $\int_{\gamma_3} x \, dz = \int_0^1 (1 - t)(-dt) = -1/2$.

γ_4: $\gamma_3(t) = (1 - t)i$, with $0 \le t \le 1$. So, $\int_{\gamma_4} x \, dz = \int_0^1 0 \, (-dt) = 0$.

Addding the together: the integral over the square is i.

(b) *Compute* $\int_C \dfrac{1}{|z|} \, dz$, *where C is the unit circle.*

Solution: Parametrize the circle, as usual, by $\gamma(\theta) = e^{i\theta}$. Since $|\gamma(\theta)| = 1$ the integral is

$$\int_C \frac{1}{|z|} \, dz = \int_0^{2\pi} i e^{i\theta} \, d\theta = 0.$$

(c) *Compute* $\int_C z \cos(z^2) \, dz$, *where C is the unit circle.*

Solution: Since $z \cos(z^2)$ is entire, it is analytic on and inside the closed curve C. Therefore by Cauchy's theorem, the integral is 0.

(d) *Draw the region* $\mathbf{C} - \{x + i \sin(x) \text{ for } x \ge 0\}$. *Is this region simply connected? Could you define a branch of* log *on this region?*

Solution: Yes, the region is simply connected. Yes, you can define a branch of log on this region: To define a branch of log you have to have a region where the argument is well defined and continuous. You can do this as long as the cut blocks any path that circles the origin. The figure below illustrates values of $\arg(z)$ at a few points in the region.
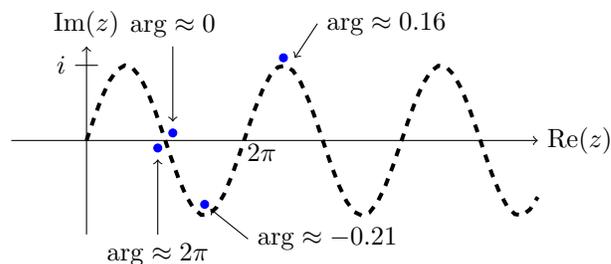
(e) *Compute $\int_C \frac{z^2}{z^4-1}$ over the circle of radius 3 with center 0.*

Solution: The fourth roots of 1 are $\pm 1, \pm i$. Thus,

$$f(z) = \frac{z^2}{z^4-1} = \frac{z^2}{(z-1)(z+1)(z-i)(z+i)}.$$

Since the curve contains all four roots we need to write it as four loops each containing just one of the roots. Then we use Cauchy's formula to compute the integral over each loop.

Loop around 1: Let $g(z) = \frac{z^2}{(z+1)(z-i)(z+i)}$. The integral of f over this loop equals $2\pi i g(1) = \pi i/2$.

Loop around -1: The integral of f over this loop is $-\pi i/2$.

Loop around i: The integral of f over this loop is $\pi/2$.

Loop around -i: The integral of f over this loop is $-\pi/2$.

Summing all 4 contributions we get 0.

(f) *Does $\int_C \frac{e^z}{z^2}\, dz = 0$?. Here C is a simple closed curve.*

Solution: Not always. We know $f(z) = e^z$ is entire. So, if C goes around 0 then, by Cauchy's formula for derivatives

$$\int \frac{f(z)}{z^2}\, dz = 2\pi i f'(0) = 2\pi i.$$

If C does not go around 0 then the integral is 0.

(g) *Compute $\int_{-\infty}^{\infty} \frac{1}{x^4+16}\, dx.$*

Solution: Let $f(z) = 1/(z^4+16)$ and let I be the integral we want to compute. The trick is to integrate f over the closed contour $C_1 + C_R$ shown, and then show that the contribution of C_R to this integral vanishes as R goes to ∞.
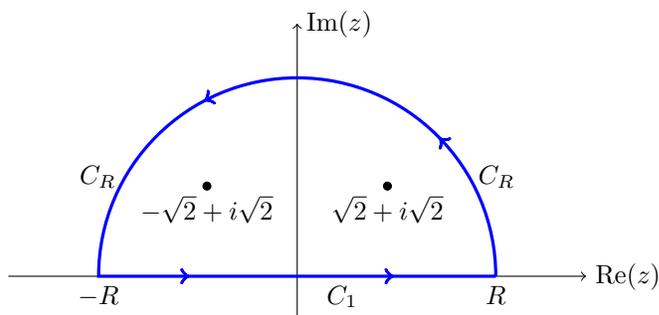
The 4 singularities of $f(z)$ are $2e^{i(\pi/4+n\pi/2)} = \pm\sqrt{2}\pm i\sqrt{2}$. The ones inside the contour are $2e^{i\pi/4} = \sqrt{2}+i\sqrt{2}$, $2e^{3i\pi/4} = -\sqrt{2}+i\sqrt{2}$. As usual we break $C_1 + C_R$ into two loops, one surrounding each singularity and use Cauchy's formula to compute the integral over each loop separately. Factoring, we have

$$f(z) = \frac{1}{z^4+16} = \frac{1}{(z-(\sqrt{2}+i\sqrt{2}))(z-(\sqrt{2}-i\sqrt{2}))(z-(-\sqrt{2}+i\sqrt{2}))(z-(-\sqrt{2}-i\sqrt{2}))}.$$

Loop around $\sqrt{2}+i\sqrt{2}$: Let $f_1(z) = \frac{1}{(z-(\sqrt{2}-i\sqrt{2}))(z-(-\sqrt{2}+i\sqrt{2}))(z-(-\sqrt{2}-i\sqrt{2}))}$. By Cauchy's integral

formula the integral is $2\pi i f_1(\sqrt{2}+i\sqrt{2}) = \dfrac{\sqrt{2}\pi(1-i)}{32}$.

Loop around $-1+i$: the integral is $\dfrac{\sqrt{2}\pi(1+i)}{32}$.

Summing, the integral around $C_1 + C_R$ is $\sqrt{2}\pi/16$.

Now we'll look at C_1 and C_r separately:

Parametrize C_1 by $\gamma(x) = x$, with $-R \le x \le R$. So

$$\int_{C_1} f(z)\,dz = \int_{-R}^{R} \frac{1}{x^4+16}\,dx.$$

This goes to the I as $R \to \infty$.

We parametrize C_R by $\gamma(\theta) = Re^{i\theta}$, with $0 \le \theta \le \pi$. So

$$\int_{C_R} f(z)\,dz = \int_0^{\pi} \frac{1}{R^4 e^{4i\theta}+16}\,iRe^{i\theta}\,d\theta /$$

By the triangle inequality, if $R > 1$

$$\left| \int_{C_R} f(z)\,dz \right| \le \int_0^{\pi} \frac{R}{R^4-16}\,d\theta = \frac{\pi R}{R^4-16}$$

Clearly this goes to 0 as R goes to infinity.

Thus, the integral over the contour $C_1 + C_R$ goes to I as R gets large. But this integral always has the same value $\sqrt{2}\pi/16$. We have shown that $I = \sqrt{2}\pi/16$.

As a sanity check, we note that our answer is real and positive as it needs to be.

Problem 6.

Suppose $f(z)$ is entire and $|f(z)| > 1$ for all z. Show that f is a constant.

Answer. Since $|f(z)| > 1$ we know f is never 0. Therefore $1/f(z)$ is entire and $|1/f(z)| < 1$. Being entire and bounded it is constant by Liouville's theorem.

Problem 7.

Suppose $f(z)$ is analytic and $|f|$ is constant on the disk $|z - z_0| \le r$. Show that f is constant on the disk.

Answer. This follows from the maximum modulus principle. Since $|f|$ is constant on the disk, its maximum modulus does not occur only on the boundary. Therefore it must be constant.

Extra problems from pset 4

Problem 8. (a) *Let $f(z) = e^{\cos(z)}z^2$. Let A be the disk $|z - 5| \le 2$. Show that $f(z)$ attains both its maximum and minimum modulus in A on the circle $|z - 5| = 2$.*

Hint: Consider $1/f(z)$.

Solution: Since $f(z)$ is analytic on and inside the disk, the maximum modulus principle tells us it attains its maximum modulus on the boundary.

Since e^w is never 0 and z^2 is not zero anywhere in A we know that $1/f(z)$ is analytic on and inside the disk. Therefore it attains its maximum modulus on the boundary. But the point where $1/|f(z)|$ is maximized is the point where $|f(z)|$ is minimized.

(b) *Suppose $f(z)$ is entire. Show that if $f^{(4)}(z)$ is bounded in the whole plane then $f(z)$ is a polynomial of degree at most 4.*

Solution: By the maximum modulus principle $f^{(4)}(z)$ is a constant. Integrating a constant 4 times leads to a polynomial of degree a most 4.

(c) *The function $f(z) = 1/z^2$ goes to 0 as $z \to \infty$, but it is not constant. Does this contradict Liouville's theorem?*

Solution: No, Liouville's theorem requires the function be entire. $f(z)$ has a singularity at the origin, so it is not entire.

Problem 9.

Show $\int_0^\pi e^{\cos\theta} \cos(\sin(\theta)) \, d\theta = \pi$. Hint, consider e^z/z over the unit circle.

Solution: (Follow the hint.) Parametrize the unit circle as $\gamma(\theta) = e^{i\theta}$, with $0 \le \theta \le 2\pi$. So,

$$\int_\gamma \frac{e^z}{z} \, dz = \int_0^{2\pi} \frac{e^{\cos\theta + i\sin\theta}}{e^{i\theta}} i e^{i\theta} \, d\theta = i \int_0^{2\pi} e^{\cos\theta + i\sin} \, d\theta$$

$$= i \int_0^{2\pi} e^{\cos\theta} (\cos(\sin\theta) + i\sin(\sin\theta)) \, d\theta = \int_0^{2\pi} e^{\cos\theta} (i\cos(\sin\theta) - \sin(\sin\theta)) \, d\theta.$$

This is close to what we want. Let's use Cauchy's integral formula to evaluate it and then extract the

value we need. By Cauchy the integral is $2\pi i e^0 = 2\pi i$. So,

$$\int_0^{2\pi} e^{\cos\theta}(i\cos(\sin\theta) - \sin(\sin\theta))\,d\theta = 2\pi i.$$

Taking the imaginary part we have

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta = 2\pi.$$

This integral is 2π, while our integral is supposed to be π. But, by symmetry ours is half the above. (It might be easier to see this if you use the limits $[-\pi, \pi]$ instead of $[0, 2\pi]$.)

So, we have shown that the integral is π.

Problem 10.

(a) *Suppose $f(z)$ is analytic on a simply connected region A and γ is a simple closed curve in A.. Fix z_0 in A, but not on γ. Use the Cauchy integral formulas to show that*

$$\int_\gamma \frac{f'(z)}{z - z_0}\,dz = \int_\gamma \frac{f(z)}{(z - z_0)^2}\,dz.$$

Since A is simply connected we know f and f' are analytic on and inside γ. Therefore we can use Cauchy's formulas.

$$\int_\gamma \frac{f'(z)}{z - z_0}\,dz = 2\pi i f'(z_0) \qquad \text{(by Cauchy's integral formula.)}$$

$$\int_\gamma \frac{f(z)}{(z - z_0)^2}\,dz. = 2\pi i f'(z_0) \qquad \text{(by Cauchy's formula for derivatives.)}$$

These are the same, so we are done.

(b) *Challenge: Redo part (a), but drop the assumption that A is simply connected.*

Let $g(z) = \frac{f(z)}{z - z_0}$. g is analytic on a neighborhood of γ. Note: $g'(z) = \frac{f'(z)}{z - z_0} - \frac{f(z)}{(z - z_0)^2}$. So,

$$\int_\gamma \frac{f'(z)}{z - z_0}\,dz - \int_\gamma \frac{f(z)}{(z - z_0)^2}\,dz = \int_\gamma g'(z)\,dz = 0.$$

It equals 0 because the integral of a derivative around a closed curve is 0. So, the two integrals on the left side are equal.

Problem 11.

(a) *Compute* $\int_C \frac{\cos(z)}{z}\,dz$, *where C is the unit circle.*

Solution: $2\pi i \cos(0) = 2\pi i$.

(b) *Compute* $\int_C \frac{\sin(z)}{z}\,dz$, *where C is the unit circle.*

Solution: $2\pi i \sin(0) = 0$.

(c) *Compute* $\int_C \dfrac{z^2}{z-1} \, dz$*, where C is the circle* $|z| = 2$.

Solution: $2\pi i z^2|_{z=1} = 2\pi i$.

(d) *Compute* $\int_C \dfrac{e^z}{z^2} \, dz$*, where C is the circle* $|z| = 1$.

Solution: $2\pi i \left.\dfrac{de^z}{dz}\right|_{z=0} = 2\pi i$.

(e) *Compute* $\int_C \dfrac{z^2-1}{z^2+1} \, dz$*, where C is the circle* $|z| = 2$.

Solution: Singularities are at $\pm i$.

$$\text{Integral } = 2\pi i \frac{-2}{2i} + 2\pi i \frac{-2}{-2i} = 0.$$

(f) *Compute* $\int_C \dfrac{1}{z^2+z+1} \, dz$ *where C is the circle* $|z| = 2$.

Solution: There are two roots. Splitting the contour as we've done several times leads to a total integral of 0.

Problem 12.

Suppose $f(z)$ is entire and $\lim\limits_{z\to\infty} \dfrac{f(z)}{z} = 0$*. Show that $f(z)$ is constant.*

You may use Morera's theorem: if $g(z)$ is analytic on $A - \{z_0\}$ and continuous on A, then f is analytic on A.

Solution: Let $g(z) = \dfrac{f(z) - f(0)}{z}$. Since $g(z)$ is analytic on $\mathbf{C} - \{0\}$ and continuous on $\mathbf{C}$ it is analytic on all of $\mathbf{C}$, by Morera's theorem

We claim $g(z) \equiv 0$.

Suppose not, then we can pick a point z_0 with $g(z_0) \neq 0$. Since $g(z)$ goes to 0 as $|z|$ gets large we can pick R large enough that $|g(z)| < |g(z_0)|$ for all $|z| = R$. But this violates the maximum modulus theorem, which says that the maximum modulus of $g(z)$ on the disk $|z| \leq R$ occurs on the circle $|z| = R$. This disaster means our assumption that $g(z) \neq 0$ was wrong. We conclude $g(z) \equiv 0$ as claimed.

This means that $f(z) = f(z_0)$ for all z, i.e. $f(z)$ is constant.

18.04 Complex Variables with Applications

Spring 2018