18.04 Recitation 13
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1. Compute $\mathcal{L}(\sin(\omega t); s)$, where $\omega \in \mathbb{R}$. For which values of s can you do this?

Ans: $\sin(\omega t) = (e^{i\omega t} - e^{-i\omega t})/2i$. Since $\mathcal{L}$ is linear, we get that

$$\begin{aligned}
\mathcal{L}(\sin(\omega t); s) &= \frac{1}{2i} \times \left(\mathcal{L}(e^{i\omega t}; s) - \mathcal{L}(e^{-i\omega t}; s)\right) \\
&= \frac{1}{2i} \times \left(\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\right) \\
&= \frac{\omega}{s^2 + \omega^2}
\end{aligned}$$

where the second line is valid for $\mathrm{Re}(s) > 0$.

2. Suppose $f(t)$ has exponential type a. Show that $\mathcal{L}(f'; s) = s\mathcal{L}(f; s) - f(0)$ for any s with $\mathrm{Re}(s) > a$. Use this to show that $\mathcal{L}(f''; s) = s^2\mathcal{L}(f; s) - sf(0) - f'(0)$ for any s with $\mathrm{Re}(s) > a$, provided that $f'(t)$ also has exponential type a.

Ans:

$$\begin{aligned}
\mathcal{L}(f'; s) &= \int_0^\infty f'(t)e^{-st}dt \\
&= fe^{-st}|_0^\infty + s\int_0^\infty f(t)e^{-st}dt \\
&= -f(0) + s\mathcal{L}(f; s)
\end{aligned}$$

where the last line uses the assumption that $\mathrm{Re}(s) > a$. Assuming that f' also has exponential type a, we get that

$$\begin{aligned}
\mathcal{L}(f'', s) &= -f'(0) + s\mathcal{L}(f'; s) \\
&= -f'(0) + s\left(-f(0) + s\mathcal{L}(f; s)\right) \\
&= s^2\mathcal{L}(f; s) - sf(0) - f'(0).
\end{aligned}$$

3. Suppose that $f(t)$ has exponential type a, and $\mathrm{Re}(s) > a$. Show that $\mathcal{L}(tf(t); s) = -\frac{d}{ds}\mathcal{L}(f(t); s)$. Use this to find $\mathcal{L}(t^n; s)$ for all integers $n \geq 0$ for $\mathrm{Re}(s) > 0$.

Ans:

$$\begin{aligned}
\frac{d}{ds}\mathcal{L}(f(t); s) &= \frac{d}{ds}\int_0^\infty f(t)e^{-st}ds \\
&= -\int_0^\infty tf(t)e^{-st}ds \\
&= -\mathcal{L}(tf(t); s).
\end{aligned}$$

We know that $\mathcal{L}(1; s) = 1/s$. Therefore,

$$\begin{aligned}
\mathcal{L}(t; s) &= \frac{1}{s^2} \\
\mathcal{L}(t^2; s) &= \frac{2}{s^3} \\
\mathcal{L}(t^3; s) &= \frac{3 \times 2}{s^4} \\
\mathcal{L}(t^4; s) &= \frac{2 \times 3 \times 4}{s^5}
\end{aligned}$$

and so on. Continuing this way, we see that $\mathcal{L}(t^n; s) = \frac{n!}{s^{n+1}}$.

4. Explain why the following pairs of functions have the same Laplace transform.

4.1. $f(t) = 1$ for all t; $u(t)$ defined by $u(t) = 1$ if $t > 0$ and $u(t) = 0$ if $t < 0$.

Ans: In the definition of the Laplace transform, we only integrate over t from 0 to ∞. For this region, the values of f and u are the same, therefore their Laplace transforms are also the same.

4.2. $f(t) = e^{at}$ for all t; $g(t)$ defined by $g(t) = e^{at}$ if $t \neq 2$ and $g(t) = 0$ if $t = 2$.

Ans: Fix s, and note that $h(t) := f(t)e^{-st} - g(t)e^{-st}$ satisfies $h(t) = 0$ if $t \neq 2$ and $h(t) = e^{(a-s)t}$ if $t = 2$. Since the integral of any function which is nonzero at just one point is 0, we get that $\int_0^\infty h(t)dt = 0$ i.e. $\int_0^\infty f(t)e^{-st} = \int_0^\infty g(t)e^{-st}$ (whenever one of these integrals converges).

5. Use the Laplace transform and partial fractions to solve the differential equation

$$x'' + 8x' + 7x = e^{-2t}$$

with initial conditions $x(0) = 0, x'(0) = 1$.

Ans: Let $X(s) := \mathcal{L}(x(t); s)$. Then, taking the Laplace transform of both sides of the differential equation, we get

$$\mathcal{L}(x''(t); s) + 8\mathcal{L}(x'(t); s) + 7\mathcal{L}(x(t); s) = \mathcal{L}(e^{-2t}; s)$$

i.e.

$$\{s^2X - sx(0) - x'(0)\} + 8\{sX - x(0)\} + 7X = \frac{1}{s+2}$$

i.e.

$$X(s^2 + 8s + 7) = \frac{1}{s+2} + 1.$$

Thus,

$$\begin{aligned}
X &= \frac{1}{(s+2)(s+7)(s+1)} + \frac{1}{(s+7)(s+1)} \\
&= \frac{-1/5}{(s+2)} + \frac{1/30}{(s+7)} + \frac{1/6}{(s+1)} + \frac{-1/6}{(s+7)} + \frac{1/6}{(s+1)} \\
&= \frac{-1/5}{(s+2)} + \frac{1/3}{(s+1)} + \frac{-2/15}{(s+7)}.
\end{aligned}$$

So,

$$X = \frac{1}{3}e^{-t} - \frac{1}{5}e^{-2t} - \frac{2}{15}e^{-7t}.$$