18.04 Recitation 3
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1. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. We write $f(x, y) = u(x, y) + iv(x, y)$. Suppose that u and v are C^2 i.e. all partial derivatives of u and v of order up to (and including) 2 exist, and are continuous. Show that $f' = \frac{df}{dz} : \mathbb{C} \to \mathbb{C}$ is also analytic.

Ans: $f' = u_x + iv_x$. Let $U = u_x$, $V = v_x$. Then, $U_x = u_{xx}$, $U_y = u_{xy}$, $V_x = v_{xx}$, $V_y = v_{xy}$. Therefore, $U_x = u_{xx} = (v_y)_x = v_{yx} = v_{xy} = V_y$. Also, $U_y = u_{xy} = u_{yx} = -v_{xx} = -V_x$.

2.1. Show that $\int \bar{z}dz$ is not path independent in $\mathbb{C}$. Why does this not contradict the fundamental theorem for complex line integrals?

Ans: Consider $\int_\gamma \bar{z}dz$ where γ is the unit circle centered at the origin. Parameterizing γ by $\gamma(t) = e^{it}$, we get $\int_\gamma \bar{z}dz = \int_0^{2\pi} e^{-it}ie^{it}dt = 2\pi i$.

2.2. For each $n \in \mathbb{Z}$, compute $\int_\gamma z^n dz$, where γ is the unit circle centered at the origin. Are your answers consistent with the fundamental theorem?

Ans: For $n \geq 0$, this is 0 by the fundamental theorem since $z^n = \frac{1}{n+1}\frac{d}{dz}z^{n+1}$ on $\mathbb{C}$.

For $n < -1$, this is 0 by the fundamental theorem since $z^n = \frac{1}{n+1}\frac{d}{dz}z^{n+1}$ on $\mathbb{C}\backslash\{0\}$, and γ is completely contained in $\mathbb{C}\backslash\{0\}$.

For $n = -1$, parameterize γ by $\gamma(t) = e^{it}$ to get $\int_\gamma z^{-1}dz = \int_0^{2\pi} e^{-it}ie^{it}dt = 2\pi i$.

2.3. Do any of the answers in 2.2. change if γ is a circle such that the disk bounded by the circle does not contain the origin?

Ans: All the answers are now zero, since we can enclose a circle not containing the origin in a region where $\log(z)$ is analytic, and then we can use that $z^{-1} = \frac{d}{dz}\log(z)$ in such a region.

3. Recall from Recitation 2 that $\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$.

3.1. Consider the region $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : 0 < x < \pi\}$. What are the images of horizontal and vertical lines in $\mathcal{R}$? Is the mapping $z \mapsto \cos(z)$ restricted to $\mathcal{R}$ a one-to-one mapping?

Ans: Vertical lines $x_0 + it$ are sent to $\cos(x_0 + it) = \cos(x_0)\cosh(t) - i\sin(x_0)\sinh(t)$.

Viewed as a map to $\mathbb{R}^2$, this is $(\cos(x_0)\cosh(t), -\sin(x_0)\sinh(t))$. This satisfies

$$\frac{u^2}{\cos^2(x_0)} - \frac{v^2}{\sin^2(x_0)} = 1$$

which is the equation of a hyperbola.

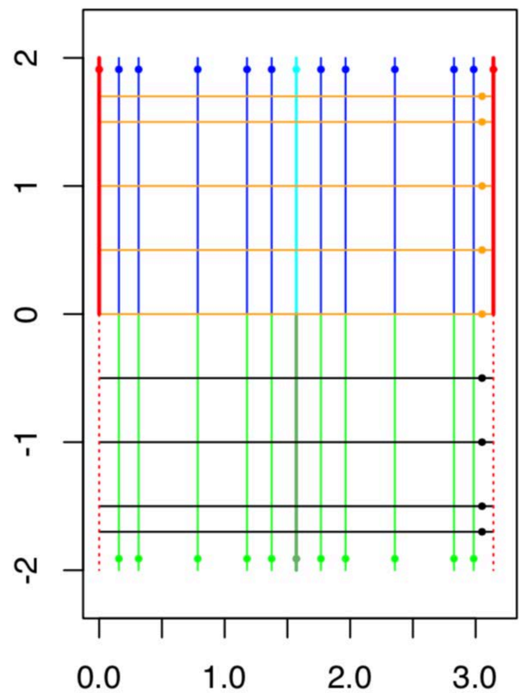
Horizontal lines $t + iy_0$ are sent to $\cos(t + iy_0) = \cos(t)\cosh(y_0) - i\sin(t)\sinh(y_0)$.

This satisfies

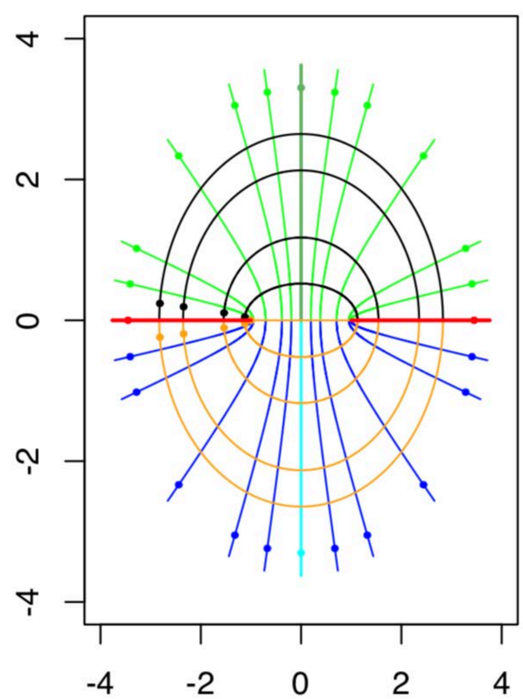
$$\frac{u^2}{\cosh^2(y_0)} + \frac{v^2}{\sinh^2(y_0)} = 1$$

which is the equation of an ellipse.

See attached figure.



$z = x + iy$



$w = \cos(z)$

3.2. To $\mathcal{R}$, add the half lines $x = 0, y \geq 0$ and $x = \pi, y > 0$ to produce a new region $\mathcal{R}_1$. What is the image of $\mathcal{R}_1$ under the map $z \mapsto \cos(z)$? Is the map still one-to-one on $\mathcal{R}_1$?

Ans: The image consists of the entire complex plane. Yes, the map is one-to-one.

3.3. Note that $\mathcal{R}_1$ gives a branch of the multi-valued function $\cos^{-1}(z)$. What are the branch cuts in the domain of $\cos^{-1}(z)$ for this branch?

Ans: $(-\infty, -1) + i0$ and $(1, \infty) + i0$. Note that as you cross these segments vertically (say from up to down), $\cos^{-1}(z)$ jumps sharply.