18.04 Recitation 9
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1. Evaluate $I_1 = \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx$.
Ans: Example 9.4 in the notes.

2. Evaluate $I_2 = \int_0^{2\pi} \frac{1}{2-\sin\theta} d\theta$.
Ans: We begin by writing $z = e^{i\theta}$, so that $dz = ie^{i\theta} d\theta = izd\theta$. Also, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$. Making these substitutions, we get

$$I_2 = \int_{|z|=1} \frac{2i}{4i - \left(z - z^{-1}\right)} \frac{dz}{iz} = \int_{|z|=1} \frac{2}{4iz - (z^2-1)} dz.$$

Let

$$f(z) = -\frac{1}{z^2 - 4iz - 1} = -\frac{1}{(z - i(2+\sqrt{3}))(z - i(2-\sqrt{3}))}.$$

Then,

$$\begin{aligned} I_2 &= 2\int_{|z|=1} f(z)dz \\ &= 2 \times 2\pi i \times \sum \text{residues of } f \text{ inside the unit circle.} \end{aligned}$$

f has simple poles at $i(2+\sqrt{3})$ and $i(2-\sqrt{3})$. Only the second one is inside the unit circle. Moreover, the residue at this pole is equal to $-\frac{1}{(i(2-\sqrt{3})-i(2+\sqrt{3}))} = \frac{1}{2i\sqrt{3}}$. Therefore, $I_2 = 2\pi/\sqrt{3}$.

3. Evaluate $I_3 = \int_0^{2\pi} (\sin\theta)^{2n} d\theta$.
Ans: Making the same substitutions as for I_2, we get that

$$I_3 = \int_{|z|=1} \frac{(z^2-1)^{2n}}{2^{2n} i^{2n+1} z^{2n+1}} dz.$$

Let

$$f(z) = \frac{(z^2-1)^{2n}}{2^{2n} i^{2n+1} z^{2n+1}}.$$

Then, by the residue theorem,

$$I_3 = 2\pi i \sum \text{residues of } f \text{ inside the unit circle.}$$

$f(z)$ has a pole of order $2n+1$ at 0. The residue at this pole is given by

$$\frac{h^{(2n)}(0)}{2^{2n}i^{2n+1}(2n)!},$$

where $h(z) = (z^2-1)^{2n}$. Since $h^{(2n)}(0)/(2n)!$ is precisely the coefficient of z^{2n}, the binomial theorem gives that $h^{(2n)}(0)/(2n)! = \binom{2n}{n}(-1)^n = \binom{2n}{n}i^{2n}$.

From this, it follows that the residue is equal to

$$\frac{\binom{2n}{n}i^{2n}}{2^{2n}i^{2n+1}} = \frac{(2n)!}{(n!)^2 2^{2n} i} = \frac{(2n)!}{(2^n n!)^2 i},$$

and hence,

$$I_3 = \frac{2\pi(2n)!}{(2^n n!)^2}.$$

4. Evaluate $I_4 = \int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$.

Ans: Example 9.8 in the notes.