18.04 Practice Laplace transform, Spring 2018 Solutions

On the final exam you will be given a copy of the Laplace table posted with these problems.

Problem 1.
Do each of the following directly from the definition of Laplace transform as an integral.

(a) *Compute the Laplace transform of* $f_1(t) = e^{at}$.

(b) *Compute the Laplace transform of* $f_2(t) = t$.

(c) *Let* $F(s) = \mathcal{L}(f; s)$. *Prove the s-derivative rule:* $\mathcal{L}(tf(t); s) = -F'(s)$.

(a) $\mathcal{L}(f_1; s) = \int_0^\infty e^{at}e^{-st}\,ds = \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty = \begin{cases} 1/(s-a) & \text{if } \mathrm{Re}(s) > \mathrm{Re}(a) \\ \text{divergent} & \text{if } \mathrm{Re}(a) \le \mathrm{Re}(s) \end{cases}$

Of course, we can analytically continue this to the region $\mathbf{C} - \{a\}$.

(b) $\mathcal{L}(f_2; s) = \int_0^\infty te^{-st}\,ds = \left.\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_0^\infty = \begin{cases} 1/s^2 & \text{if } \mathrm{Re}(s) > \mathrm{Re}(0) \\ \text{divergent} & \text{if } \mathrm{Re}(0) \le \mathrm{Re}(s) \end{cases}$

Of course, we can analytically continue this to the region $\mathbf{C} - \{0\}$.

(c) (See the topic 12 notes.)

$$F'(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty -tf(t)e^{-st}\,dt = \mathcal{L}(-tf(t); s).$$

Problem 2.
For each of the following you can use the Laplace table if it helps.

(a) *Compute the Laplace transform of* $\cosh(at)$.

(b) *Compute the Laplace transform of* $f(t) = \begin{cases} 0 & \textit{for } t < 5 \\ \cosh(a(t-5)) & \textit{for } t > 5 \end{cases}$.

(c) *Compute the Laplace transform of* $f(t) = \begin{cases} \sin(t) & \textit{for } 0 \le t \le \pi \\ 0 & \textit{for } t > \pi \end{cases}$

(d) *Compute the Laplace transform of* $t\cos(at)$.

(e) *Let* $\Gamma(z) = \mathcal{L}(t^{z-1}; s = 1)$. *Show that* $\Gamma(z+1) = z\Gamma(z)$

(a) $\cosh(at) = \frac{e^{at} + e^{-at}}{2}$. So,

$$\mathcal{L}(\cosh(at); s) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}$$

(b) The t-shift rule says $\mathcal{L}(f; s) = e^{-5s}\mathcal{L}(\cosh(at); s) = \boxed{e^{-5s}\frac{s}{s^2 - a^2}}$.

(c) We could do this directly from the integrals. Instead, notice that for $t > 0$

$$f(t) = \sin(t) + \begin{cases} 0 & \text{for } t < \pi \\ \sin(t - \pi) & \text{for } t > \pi \end{cases}$$

So, $\mathcal{L}(f;s) = \mathcal{L}(\sin(t);s) + e^{-\pi s}\mathcal{L}(\sin(t);s) = \boxed{(1+e^{-\pi s})\frac{1}{s^2+1}}$.

(d) Let $F(s) = \mathcal{L}(\cos(at);s) = \frac{s}{s^2+a^2}$. The t-derivative rule says

$$\mathcal{L}(t\cos(at);s) = -F'(s) = \frac{s^2-a^2}{(s^2+a^2)^2}$$

(e) Assume $\mathrm{Re}(z) > 0$. Let $f(t) = t^z$. Since $f(0) = 0$, the t-derivative rule implies

$$\mathcal{L}(f'(t);s) = s\mathcal{L}(f(t);s) = s\mathcal{L}(t^z;s).$$

Since $f'(t) = zt^{z-1}$, we also have $\mathcal{L}(f'(t);s) = z\mathcal{L}(t^{z-1};s)$.

Thus $z\mathcal{L}(t^{z-1};s) = s\mathcal{L}(t^z;s)$. The result now follows by setting $s = 1$.

Problem 3.

(a) *Use the Laplace transform to solve the differential equation $x' + x = te^{2t}$, with $x(0) = 3$.*

Find the Laplace inverse using the formula involving the sums of residues. (Be sure to verify that the hypotheses of the theorem hold.)

First, the Laplace transform of $te^{2t} = -\frac{d}{ds}\left(\frac{1}{s-2}\right) = \frac{1}{(s-2)^2}$.

Now, take the Laplace transform of the differential equation

$$(sX(s) - x(0)) + X(s) = \frac{1}{(s-2)^2} \Rightarrow X(s) = \frac{1}{(s+1)(s-2)^2} + \frac{3}{x+1}.$$

Since $X(s)$ decays faster than $1/s$ we can apply this inversion formula using the sums of the residues.

$$x(t) = \sum \text{residues of } e^{st}X(s).$$

We split $e^{st}X(s)$ into two pieces $F_1(s) = \frac{e^{st}}{(s-2)^2(s+1)}$ and $F_2(s) = \frac{3}{s+1}$. F_1 has poles at $s = 2$ and $s = -1$.

At $s = 2$: Let $G(s) = (s-2)^2F_1(s) = \frac{e^{st}}{s+1}$. $G(s)$ is analytic at $s = 2$, so $G(s) = G(2) + G'(2)(s-2) + \ldots$. Thus,

$$\mathrm{Res}(F_1, 2) = G'(2) = \frac{te^{2t}3 - e^{2t}}{3^2} = \frac{te^{2t}}{3} - \frac{e^{2t}}{9}.$$

At $s = -1$: $\mathrm{Res}(F_1, -1) = \lim_{s\to-1}(s+1)F_1(s) = \frac{e^{-t}}{9}$.

So, $\mathcal{L}^{-1}(F_1;t) = \mathrm{Res}(F_1,2) + \mathrm{Res}(F_1,-1) = \frac{te^{2t}}{3} - \frac{e^{2t}}{9} + \frac{e^{-t}}{9}$.

Clearly, $\mathcal{L}^{-1}(F_2;t) = 3e^{-t}$. Thus, $\boxed{x(t) = \frac{te^{2t}}{3} - \frac{e^{2t}}{9} + \frac{e^{-t}}{9} + 3e^{-t}.}$

(b) *Solve $y' - y = \begin{cases} 0 & \text{for } t < 1 \\ 1 & \text{for } t > 1 \end{cases}$, with $y(0) = 0$.*

Why does the inversion formula involving sums of residues not apply?

Let $f(t) = \begin{cases} 0 & \text{for } t < 1 \\ 1 & \text{for } t > 1 \end{cases}$. Since $\mathcal{L}(1; s) = 1/s$, the t-shift rule shows that $\mathcal{L}(f; s) = \mathrm{e}^{-s}/s$.

Now take the Laplace transform of the differential equation:

$$(s-1)Y(s) = \frac{\mathrm{e}^{-s}}{s} \;\Rightarrow\; Y(s) = \frac{\mathrm{e}^{-s}}{s(s-1)}$$

Using partial fractions we find that

$$\frac{1}{s(s-1)} = -\frac{1}{s} + \frac{1}{s-1}, \qquad \text{so} \qquad \mathcal{L}^{-1}\left(\frac{1}{s(s-1)}; t\right) = -1 + \mathrm{e}^{t}.$$

Now using the t-shift rule we have

$$y(t) = \begin{cases} 0 & \text{for } t < 1 \\ -1 + \mathrm{e}^{t-1} & \text{for } t > 1. \end{cases}$$

Problem 4.
Use the Laplace transform to solve the differential equation $x'' + x = \sin(t)$, with $x(0) = 0$, $x'(0) = 0$. (Hint: use the table to do the Laplace inverse.)

The zero initial conditions make taking the Laplace transform of the differential equation easy

$$(s^2+1)X(s) = \frac{1}{s^2+1} \;\Rightarrow\; X(s) = \frac{1}{(s^2+1)^2}.$$

This is in our Laplace table. So, $\boxed{x(t) = \frac{1}{2}\left(\sin(t) - t\cos(t)\right).}$
