An overview of key ideas

This is an overview of linear algebra given at the start of a course on the mathematics of engineering.

Linear algebra progresses from vectors to matrices to subspaces.

Vectors

What do you do with vectors? Take combinations.

We can multiply vectors by scalars, add, and subtract. Given vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ we can form the *linear combination* $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{b}$.

An example in $\mathbb{R}^3$ would be:

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The collection of all multiples of $\mathbf{u}$ forms a line through the origin. The collection of all multiples of $\mathbf{v}$ forms another line. The collection of all combinations of $\mathbf{u}$ and $\mathbf{v}$ forms a plane. Taking *all combinations* of some vectors creates a *subspace*.

We could continue like this, or we can use a matrix to add in all multiples of $\mathbf{w}$.

Matrices

Create a matrix A with vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in its columns:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

The product:

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$

equals the sum $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{b}$. The product of a matrix and a vector is a combination of the columns of the matrix. (This particular matrix A is a *difference matrix* because the components of $A\mathbf{x}$ are differences of the components of that vector.)

When we say $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{b}$ we're thinking about multiplying numbers by vectors; when we say $A\mathbf{x} = \mathbf{b}$ we're thinking about multiplying a matrix (whose columns are $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$) by the numbers. The calculations are the same, but our perspective has changed.

For any input vector $\mathbf{x}$, the output of the operation "multiplication by A" is some vector $\mathbf{b}$:

$$A\begin{bmatrix}1\\4\\9\end{bmatrix}=\begin{bmatrix}1\\3\\5\end{bmatrix}.$$

A deeper question is to start with a vector $\mathbf{b}$ and ask "for what vectors $\mathbf{x}$ does $A\mathbf{x}=\mathbf{b}$?" In our example, this means solving three equations in three unknowns. Solving:

$$A\mathbf{x}=\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_2-x_1\\x_3-x_2\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

is equivalent to solving:

$$\begin{aligned}x_1&=b_1\\x_2-x_1&=b_2\\x_3-x_2&=b_3.\end{aligned}$$

We see that $x_1=b_1$ and so x_2 must equal b_1+b_2. In vector form, the solution is:

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_1+b_2\\b_1+b_2+b_3\end{bmatrix}.$$

But this just says:

$$\mathbf{x}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix},$$

or $\mathbf{x}=A^{-1}\mathbf{b}$. If the matrix A is invertible, we can multiply on both sides by A^{-1} to find the unique solution $\mathbf{x}$ to $A\mathbf{x}=\mathbf{b}$. We might say that A represents a transform $\mathbf{x}\rightarrow\mathbf{b}$ that has an inverse transform $\mathbf{b}\rightarrow\mathbf{x}$.

In particular, if $\mathbf{b}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ then $\mathbf{x}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

The second example has the same columns $\mathbf{u}$ and $\mathbf{v}$ and replaces column vector $\mathbf{w}$:

$$C=\begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{bmatrix}.$$

Then:

$$C\mathbf{x}=\begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1-x_3\\x_2-x_1\\x_3-x_2\end{bmatrix}$$

and our system of three equations in three unknowns becomes circular.

Where before $A\mathbf{x} = \mathbf{0}$ implied $\mathbf{x} = \mathbf{0}$, there are non-zero vectors $\mathbf{x}$ for which $C\mathbf{x} = \mathbf{0}$. For any vector $\mathbf{x}$ with $x_1 = x_2 = x_3$, $C\mathbf{x} = \mathbf{0}$. This is a significant difference; we can't multiply both sides of $C\mathbf{x} = \mathbf{0}$ by an inverse to find a non-zero solution $\mathbf{x}$.

The system of equations encoded in $C\mathbf{x} = \mathbf{b}$ is:

$$\begin{aligned} x_1 - x_3 &= b_1 \\ x_2 - x_1 &= b_2 \\ x_3 - x_2 &= b_3. \end{aligned}$$

If we add these three equations together, we get:

$$0 = b_1 +_2 + b_3.$$

This tells us that $C\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ only when the components of $\mathbf{b}$ sum to 0. In a physical system, this might tell us that the system is stable as long as the forces on it are balanced.

Subspaces

Geometrically, the columns of C lie in the same plane (they are *dependent*; the columns of A are *independent*). There are many vectors in $\mathbb{R}^3$ which do not lie in that plane. Those vectors cannot be written as a linear combination of the columns of C and so correspond to values of $\mathbf{b}$ for which $C\mathbf{x} = \mathbf{b}$ has no solution $\mathbf{x}$. The linear combinations of the columns of C form a two dimensional *subspace* of $\mathbb{R}^3$.

This plane of combinations of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ can be described as "all vectors $C\mathbf{x}$". But we know that the vectors $\mathbf{b}$ for which $C\mathbf{x} = \mathbf{b}$ satisfy the condition $b_1 + b_2 + b_3 = 0$. So the plane of all combinations of $\mathbf{u}$ and $\mathbf{v}$ consists of all vectors whose components sum to 0.

If we take all combinations of:

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we get the entire space $\mathbb{R}^3$; the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^3$. We say that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ form a *basis* for $\mathbb{R}^3$.

A *basis* for $\mathbb{R}^n$ is a collection of n independent vectors in $\mathbb{R}^n$. Equivalently, a basis is a collection of n vectors whose combinations cover the whole space. Or, a collection of vectors forms a basis whenever a matrix which has those vectors as its columns is invertible.

A *vector space* is a collection of vectors that is closed under linear combinations. A *subspace* is a vector space inside another vector space; a plane through the origin in $\mathbb{R}^3$ is an example of a subspace. A subspace could be equal to the space it's contained in; the smallest subspace contains only the zero vector.

The subspaces of $\mathbb{R}^3$ are:

- the origin,
- a line through the origin,
- a plane through the origin,
- all of $\mathbb{R}^3$.

Conclusion

When you look at a matrix, try to see "what is it doing?"

Matrices can be rectangular; we can have seven equations in three unknowns. Rectangular matrices are not invertible, but the symmetric, square matrix A^TA that often appears when studying rectangular matrices may be invertible.