18.075 **Handout**: Overview of evaluation of (real) definite integrals

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We have studied 4 main categories of (real) definite integrals that can be evaluated by contour integration. Not all integrals of each category are directly amenable to evaluation by the methods shown in class.

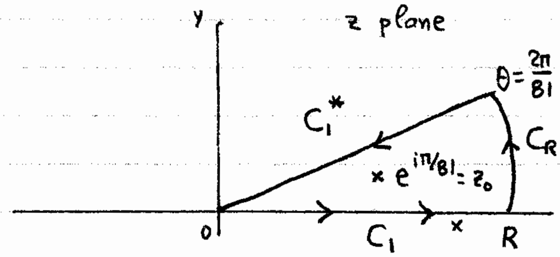
(A) Integrals of form $\int_{-\infty}^{\infty} dx\, \frac{P_n(x)}{Q_m(x)}$, $P_n(x), Q_m(x)$: n, m degree polynomials, $m \geq n+2$.

Usual sequence of steps: • Replace x by z, and locate and characterize the singularities of $\frac{P_n(z)}{Q_m(z)}$; these points are poles by $Q_m(z)=0$.

• Close the original path by a large semicircle of radius R in the upper or lower half plane (choice is immaterial); ultimately allow $R \to \infty$ so that the integral over the semicircle vanishes (by Theorem 1). Apply the Residue Theorem for poles enclosed by the total contour.

Special (sub) class of integrals: • If the integral is $\int_0^{\infty} dx\, \frac{P_n(x)}{Q_m(x)}$, $m \geq n+2$, then check whether it is possible to apply $\int_0^{\infty} dx\,(\cdots) = \frac{1}{2}\int_{-\infty}^{\infty} dx\,(\cdots)$, i.e., whether the integrand is even in x. Example: $\int_0^{\infty} \frac{dx}{1+x^{2m}}$, m: positive integer.

•• In case the above trial fails, then check whether it is possible to find a ray (line originating from 0) along which the integrand, or suitable part of it, takes the same values as those for $x>0$. If so, close the path by this (semi-infinite) line and the appropriate circular arc of radius R ($R \to +\infty$). In case there are more than one choices for this line, pick the line "closest" to the positive real axis so that only <u>1 pole</u> is enclosed by the total contour.



<u>Example</u> : $I = \displaystyle\int_0^\infty \frac{dx}{1+x^{81}}$

$(x \to z)$ $\quad \dfrac{1}{1+z^{81}}$ has simple poles at

$$1+z^{81} = 0 \iff z^{81} = e^{i\pi} \iff z = z_n = e^{(i\pi + 2n\pi)/81}, \quad n = 0, 1, 2, \ldots, 80.$$

The pole "closest" to the positive real axis is $z_0 = e^{i\pi/81}$.

Along the ray $z = \overset{|z|}{\overset{\downarrow}{x}}\, e^{i\frac{2\pi}{81}}$ $(x > 0)$, $\dfrac{1}{1+z^{81}} = \dfrac{1}{1+x^{81}}$: same values as for z: real >0.

Take $C = C_1 + C_R + C_1^*$ as shown above; C_1 is the original path for $R \to \infty$.

Residue Theorem: $$\oint_C \frac{dz}{1+z^{81}} = 2\pi i \operatorname*{Res}_{z=z_0}\left[\frac{1}{1+z^{81}}\right] = 2\pi i \frac{1}{81 z_0^{80}} = \frac{2\pi i}{81} e^{-i\pi\frac{80}{81}} \qquad (1)$$

$$\oint_C = \int_{C_1} + \int_{C_R} + \int_{C_1^*} \quad : \quad \bullet \lim_{R\to\infty} \int_{C_R} \frac{dz}{1+z^{81}} = 0, \text{ by Theorem 1.}$$

$$\bullet \int_{C_1^*} \frac{dz}{1+z^{81}} \overset{z = xe^{i\frac{2\pi}{81}}}{=\!=} -e^{i\frac{2\pi}{81}} \int_0^\infty \frac{dx}{1+x^{81}} = -e^{i\frac{2\pi}{81}} I \qquad (\text{as } R \to \infty)$$

Putting the pieces together: $$I + 0 - e^{i\frac{2\pi}{81}} I = \frac{2\pi i}{81} e^{-i\frac{80}{81}\pi}$$

$$\iff \underbrace{\left(1 - e^{i\frac{2\pi}{81}}\right)}_{-e^{+i\frac{\pi}{81}} 2i\sin\frac{\pi}{81}} I = \frac{2\pi i}{81} e^{-i\frac{80}{81}\pi} \iff -2i\sin\frac{\pi}{81}\, I = \frac{2\pi i}{81} e^{-i\frac{80\pi}{81}} e^{-i\pi/81} = -\frac{2\pi i}{81}$$

$$\iff \boxed{I = \frac{(\pi/81)}{\sin\frac{\pi}{81}}}$$

Ⓑ Integrals $\int_{-\infty}^{\infty} dx\ e^{iax}\ \frac{P_n(x)}{Q_m(x)}$, $\int_{-\infty}^{\infty} dx \left.\begin{matrix}\cos(ax)\\ \sin(ax)\end{matrix}\right\} \frac{P_n(x)}{Q_m(x)}$, a: real $\neq 0$, $m \geq n+1$.

Method : $$\int_{-\infty}^{\infty} dx \left.\begin{matrix}\cos(ax)\\ \sin(ax)\end{matrix}\right\} \frac{P_n(x)}{Q_m(x)} = \left.\begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\right\} \cdot \int_{-\infty}^{\infty} dx\ e^{iax}\ \frac{P_n(x)}{Q_m(x)} .$$

(Motive: You want an exponential to appear under the integral!)

Try to repeat the sequence of steps of Ⓐ for $\int_{-\infty}^{\infty} dx\ e^{iax}\ \frac{P_n(x)}{Q_m(x)}$.

However , close the path in the upper half plane if $a>0$, and in the lower half plane if $a<0$. (by Theorem 2).

Exceptions (case Ⓓ in class): $\frac{P_n(x)}{Q_m(x)}$ has singularities by $Q_m(x)=0$ on the real axis that are canceled by $\cos(ax)$ or $\sin(ax)$ in original integral.

Example : $\int_{-\infty}^{\infty} dx\ \frac{\sin x}{(\pi^2-x^2)x}$, $\int_{-\infty}^{\infty} dx\ \frac{\cos x}{4x^2-\pi^2}$.

In such cases , $$\int_{-\infty}^{\infty} dx \left.\begin{matrix}\cos(ax)\\ \sin(ax)\end{matrix}\right\} \frac{P_n(x)}{Q_m(x)} : \text{finite} \neq \left.\begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\right\} \int_{-\infty}^{\infty} dx\ e^{iax}\ \frac{P_n(x)}{Q_m(x)} = \infty \quad \text{(usually)}$$

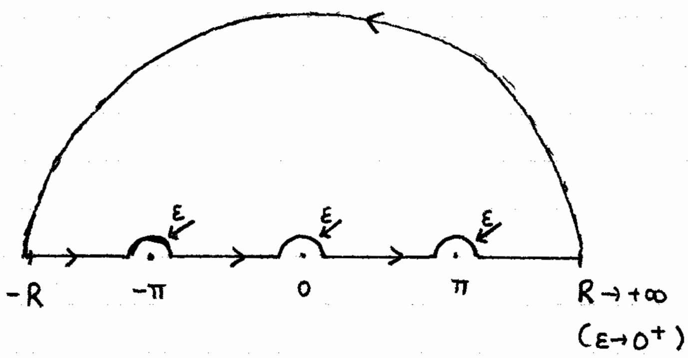
Remedy : Define principal value by taking

$$\int_{-\infty}^{\infty} dx \left.\begin{matrix}\cos(ax)\\ \sin(ax)\end{matrix}\right\} \frac{P_n(x)}{Q_m(x)} = P\int_{-\infty}^{\infty} dx \left.\begin{matrix}\cos(ax)\\ \sin(ax)\end{matrix}\right\} \frac{P_n(x)}{Q_m(x)} = \left.\begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\right\} \left(P\int_{-\infty}^{\infty} dx\ e^{iax}\ \frac{P_n(x)}{Q_m(x)} \right)$$

Calculate $P\int_{-\infty}^{\infty} dx\ e^{iax}\ \frac{P_n(x)}{Q_m(x)}$ by technique of indented contours , i.e. , by closing the path through small semicircles around the singularities of real axis.

Example : For $\int_{-\infty}^{\infty} dx\ \frac{\sin x}{x(\pi^2-x^2)} = \mathrm{Im}\, P\int_{-\infty}^{\infty} dx\ \frac{e^{ix}}{x(\pi^2-x^2)} \equiv \mathrm{Im} \lim_{\varepsilon\to 0^+} \left(\int_{-\infty}^{-\pi-\varepsilon} + \int_{-\pi+\varepsilon}^{-\varepsilon} + \int_{\varepsilon}^{\pi-\varepsilon} + \int_{\pi+\varepsilon}^{\infty} \right) dx\ \frac{e^{ix}}{x(\pi^2-x^2)}$, an indented contour is shown below.

$\left[\text{Figure for } \int_{-\infty}^{\infty} \frac{\sin x}{x(\pi^2 - x^2)} dx \right]$



Recall: Each small semicircle does contribute (as $\epsilon \to 0^+$) according to Theorem 4.

[Remark: Each small semicircle can be in the lower or upper half plane at will; same results should follow. However, one should make a particular choice from the very beginning and "stick" to it till the end of calculation. The choice of where the semicircles lie affects the application of the Residue Theorem: since the corresponding points are poles of integrand, the poles enclosed by the total path are determined by whether the small semicircle(s) lies above or below the real axis.]

By closing the path by a large semicircle in the upper $(a>0)$ or lower $(a<0)$ half plane, apply the Residue Theorem for poles enclosed by total contour. [Contribution of large semicircle should vanish by Theorem 2]

© Integrals $\int_0^{2\pi} d\theta \; F(\sin\theta, \cos\theta)$ — some rational function.

Warning: If the original integral has different limits, try to give it this form so that periodicity is evident and $\theta \in (0, 2\pi)$. Example: $\int_0^{\pi} d\theta \frac{1}{A + B\cos^2\theta}$ (hmwk)

Method: Set $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$

$$\begin{cases} \cos z = \frac{z + z^{-1}}{2} \\ \sin z = \frac{z - z^{-1}}{2i} \end{cases}$$

Then,

$$\int_0^{2\pi} d\theta \; F(\sin\theta, \cos\theta) = \oint_C \frac{dz}{iz} \; F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2} \right), \quad C: \text{unit circle}, \; |z| = 1$$

Use Residue Theorem.