18.085 Computational Science and Engineering I
Fall 2008

FALL 2002 QUIZ2 SOLUTIONS

Problem 1 (40 points)

This question is about a fixed-free hanging bar (made of 2 materials) with a point load at $x = \frac{3}{4}$:

$$\begin{aligned} -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) &= \delta\left(x - \frac{3}{4}\right) \\ u(0) &= 0 \\ w(1) &= 0 \end{aligned}$$

Suppose that

$$c(x) = \begin{cases} 1, & x < \frac{1}{2} \\ 4, & x > \frac{1}{2} \end{cases}$$

a) (i) At $x = \frac{1}{2}$, u and w are continuous. Then u_x must have a jump (ii) At $x = \frac{3}{4}$, u is continuous (as always) while w jumps by 1. We should expect $\frac{du}{dx}$ to have a jump unless such jump is "accidentally" 0.

b)

$$w(x) = \begin{cases} A, & 0 < x < \frac{3}{4} \\ B, & \frac{1}{2} < x < \frac{3}{4} \\ C, & \frac{3}{4} < x < 1 \end{cases}$$

where the three constants A, B, and C are determined from the boundary condition $w(1) = 0$, resulting in

$$C = 0,$$

continuity of w at $x = \frac{1}{2}$, resulting in

$$4B - A = 0,$$

and $[w]_{-}^{+} = -1$, resulting in

$$4C - 4B = -1$$

This system with three equation and three unkowns is easily solved, yielding $A = 1$, $B = \frac{1}{4}$, $C = 0$. Summarizing:

$$w(x) = \begin{cases} 1, & 0 < x < \frac{3}{4} \\ \frac{1}{4}, & \frac{1}{2} < x < \frac{3}{4} \\ 0, & \frac{3}{4} < x < 1 \end{cases}$$

c)

$$u = \begin{cases} x + D, & 0 < x < \frac{1}{2} \\ \frac{1}{4}x + E, & \frac{1}{2} < x < \frac{3}{4} \\ F, & \frac{3}{4} < x < 1 \end{cases}$$

where the three constants D, E, and F are detemined from the boundary condition $u(0) = 0$:

$$D = 0,$$

continuity of u at $x = \frac{1}{2}$:

$$\frac{1}{4} \times \frac{1}{2} + E - \frac{1}{2} - D = 0,$$

and continuity of u at $x = \frac{3}{4}$:

$$F - \frac{1}{4} \times \frac{3}{4} - E = 0.$$

We find that $D = 0$, $E = \frac{3}{8}$, $F = \frac{9}{16}$ and so

$$u = \begin{cases} x, \ 0 < x < \frac{1}{2} \\ \frac{1}{4}x + \frac{3}{8}, \frac{1}{2} < x < \frac{3}{4} \\ \frac{9}{16}, \ \frac{3}{4} < x < 1 \end{cases}$$

Problem 2 (30 points)

a)

(i) It is easy to show that

$$\left(\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right)(x+iy) = 1$$

Therefore,

$$\frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

and the real and imaginary parts are

$$\begin{aligned} u(x,y) &= \frac{x}{x^2+y^2} \\ s(x,y) &= \frac{-y}{x^2+y^2} \end{aligned}$$

(ii) In polar coordinates we have

$$\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$$

Therefore,

$$\begin{aligned} u(r,\theta) &= \frac{1}{r}\cos\theta \\ s(r,\theta) &= -\frac{1}{r}\sin\theta \end{aligned}$$

b) The curve curve $u(x,y) = \frac{1}{2}$ has the following equation:

$$\frac{x}{x^2+y^2} = \frac{1}{2}$$

This equation is equivalent to

$$x^2 + y^2 - 2x = 0$$

or

$$x^2 - 2x + 1 + y^2 = 1$$

or, finally

$$(x-1)^2 + y^2 = 1$$

Similarly, the curve $s(x,y) = \frac{1}{2}$ is given by

$$x^2 + (y+1)^2 = 1$$

The curve $u(x,y) = \frac{1}{2}$is a circle of radius 1 centered at the point $(1,0)$ while the curve $s(x,y) = \frac{1}{2}$ is a circle of radius 1 centered at $(0,-1)$.

c) On the part $u(x,y) = \frac{1}{2}$ simply take $u_0 = \frac{1}{2}$. Now let's look at the other part. It is orthogonal to the equipotentials of u. In other words, u does not change in the directions orthogonal to the part. Analytically, this is expressed as $\frac{\partial u}{\partial n} = 0$ or $w \cdot n = 0$.

Problem 3 (30 points)

a). We have

$$\begin{aligned}
u_x &= \frac{\partial^2 F}{\partial y \partial x} \\
u_y &= \frac{\partial^2 F}{\partial y^2} \\
s_x &= \frac{\partial^2 F}{\partial x^2} \\
s_y &= \frac{\partial^2 F}{\partial x \partial y}
\end{aligned}$$

It can be immediately observed that $u_x = s_y$ since partial derivatives commute. Also, $u_y + s_x = \Delta F = 0$ since F is harmonic.

b). The test for "having originated from a potential" is " $\text{curl}\, v$" $= \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0$. Reconstructing the potential is a little less straightforward.

(i) $v(x,y) = \left(x^2, y^2\right)$: Yes, $u = \frac{1}{3}x^3 + \frac{1}{3}y^2$, $\Delta u = 2x + 2y \neq 0$

(ii) $v(x,y) = \left(y^2, x^2\right)$: No.

(iii) $v(x,y) = (x+y, x-y)$: Yes, $u = \frac{1}{2}x^2 + xy - \frac{1}{2}y^2$, $\Delta u = 0$

c) (i)

$$u(r,\theta) = \frac{1}{2} + r\cos\theta + r^2\cos 2\theta$$

(ii)

$$u(r=0,\theta) = \frac{1}{2}$$

(A harmonic function equals to the average of its neighbors!)

$$u\left(r = \frac{1}{2}, \theta = 0\right) = \frac{5}{4}$$

Miscellaneous

Problem 1d) This is what the solution would be if we were to account for the weight P of the bottom part of the bar

$$w(x) = \begin{cases} G, \ 0 < x < \frac{3}{4} \\ -Px + H, \frac{1}{2} < x < \frac{3}{4} \\ -Px + I, \ \frac{3}{4} < x < 1 \end{cases}$$

The same three conditions will determine the constants: $w(1) = 0$:

$$I - P = 0,$$

continuity of w at $x = \frac{1}{2}$:

$$4\left(-\frac{1}{2}P + H\right) - G = 0,$$

and the jump in w at $x = \frac{3}{4}$:

$$4I - 4H = -1$$

The resulting system determines the unknown constants: $G = 6P + 1$, $H = P + \frac{1}{4}$, and $I = P$:

$$w(x) = \begin{cases} 6P + 1, \ 0 < x < \frac{3}{4} \\ P(1 - x) + \frac{1}{4}, \frac{1}{2} < x < \frac{3}{4} \\ P(1 - x), \ \frac{3}{4} < x < 1 \end{cases}$$