18.100A

True-False review questions on sequences and series

Each of the statements below is either true or false. If true, prove. If false, give a counterexample. [$\langle a_n \rangle$ is a sequence, another notation for $\{a_n\}$]
(T + F is on the bottom upside down, separated by ‖, in order (numerically))

1. Given $\epsilon > 0$, $a_n \ge 1 - \epsilon$ for $n \gg 1$ $\Rightarrow \lim_{n\to\infty} a_n = 1$

2. $\langle x_n \rangle$ converges $\Rightarrow \langle x_n \rangle$ is a Cauchy sequence.

3. a) $x_n \to 0 \Leftrightarrow |x_n| \to 0$ b) $x_n \to L \Leftrightarrow |x_n| \to |L|$

4. $\langle a_n \rangle$ positive, decreasing $\Rightarrow \sum (-1)^n a_n$ converges

5. $\langle a_n \rangle$ diverges $\Rightarrow a_n \to \infty$ or $a_n \to -\infty$

6a. $\langle x_n \rangle$ converges $\Rightarrow \langle x_n \rangle$ bounded for all n
b. $\Leftarrow$

7. $n a_n \to 0 \Rightarrow \sum a_n$ converges

8a. For all $\epsilon > 0$, $|a_{n+1} - a_n| < \epsilon$ for $n \gg 1$ $\Rightarrow \langle a_n \rangle$ converges.
b. $\Leftarrow$

9. $\langle x_n \rangle \to 0 \Leftrightarrow \left\langle \frac{1}{x_n} \right\rangle \to \infty$

10. $\xi = \sup \langle a_n \rangle \Rightarrow \xi$ is a cluster pt. of $\langle a_n \rangle$

11. If $\left| \frac{a_{n+1}}{a_n} \right| < 1$ for all n, then $\sum a_n$ converges.

12a. $\sum a_n$ converges $\Rightarrow a_n \to 0$.
b. $\Leftarrow$

13. Every power series has a positive radius of convergence R, or $R = \infty$.

14. $a_n \to L$, $L > 0 \Rightarrow a_n > 0$ for $n \gg 1$.

15. $a_n \to L$, $L \ge 0 \Rightarrow a_n \ge 0$ for $n \gg 1$.

F | T | T,F | F | F | T,F | F | F,T | F | F | F | T,F | F | T | F

1 Given $\epsilon > 0$,

$$\left|\frac{3n^2-1}{n^2+n} - 3\right| = \left|\frac{-1-3n}{n^2+n}\right| < \frac{1}{n^2+n} + \frac{3n}{n^2+n} \quad \text{(by } \Delta\neq\text{)}$$

$$< \frac{1}{n} + \frac{3}{n} = \frac{4}{n};$$

$$\frac{4}{n} < \epsilon \quad \text{if} \quad n > \frac{4}{\epsilon}.$$

2 By the ratio test,

$$\left|\frac{A_{n+1}}{A_n}\right| = \left|\frac{(2n+2)!\,x^{n+1}}{(n+1)!\,(n+2)!} \cdot \frac{n!\,(n+1)!}{(2n)!\,x^n}\right|$$

$$= \left|\frac{(2n+2)(2n+1)\,x}{(n+1)\,(n+2)}\right|$$

$$\longrightarrow 4|x| < 1 \iff |x| < 1/4 \quad \text{as } n\to\infty$$

$\therefore$ converges $\Leftarrow |x| < \frac{1}{4}$; $R = 1/4$.
diverges $\Leftarrow |x| > 1/4$

3 $a_n = \dfrac{c^n}{n!}$, $c > 1$

a) $\{a_n\}$ is decreasing for $n \gg 1$:

$$\frac{a_{n+1}}{a_n} = \frac{c^{n+1}}{(n+1)!} \cdot \frac{n!}{c^n} = \frac{c}{n+1}$$

$$\frac{c}{n+1} < 1 \quad \text{if} \quad n > c-1$$

$\therefore \{a_n\}$ decreasing if $n > c-1$.

b) $\lim\limits_{n\to\infty} a_n = 0$:

Let N be an integer, $N > 2c$.
Then by part (a), $\dfrac{a_{n+1}}{a_n} = \dfrac{c}{n+1} < \dfrac{1}{2}$ if $n \geq N$.

$\therefore a_{N+1} < \frac{1}{2} a_N$
$a_{N+2} < \frac{1}{2^2} a_N$, and in general

$$\begin{array}{ccccc} 0 & < & a_{N+k} & < & \dfrac{a_N}{2^k} \\ \downarrow & & \Downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array} \quad \text{as } k\to\infty \ \text{(Thm 3.4)}$$

$\therefore \lim\limits_{n\to\infty} a_n = \lim\limits_{k\to\infty} a_{N+k} = 0$, by the squeeze theorem.

c) You can use the ratio test to prove that $\sum\limits_0^\infty \dfrac{c^n}{n!}$ converges, and therefore $\lim\limits_{n\to\infty} \dfrac{c^n}{n!} = 0$, by the n^{th} term test for divergence.

4 Write $a = 1+h$. By the binomial thm,

$$a^n = (1+h)^n = 1 + nh + \frac{n(n-1)}{2}h^2 + \text{positive terms (since } h>0)$$

$$\therefore a^n > \frac{n(n-1)}{2}h^2$$

$$\frac{a^n}{n} > \frac{n-1}{2}h^2, \text{ which is} > \text{any given } M \text{ if } n-1 > \frac{2M}{h^2}.$$

$\therefore \dfrac{a^n}{n} \to \infty$ as $n\to\infty$, by Defn. 3.3

5 $\lim\limits_{n\to\infty} a_n^{1/n} = L > 1 \Rightarrow a_n^{1/n} > 1$ for $n \gg 1$, by the sequence loc'n theorem;
$\Rightarrow a_n > 1$, since $a_n > 0$; for $n \gg 1$
This shows $\lim\limits_{n\to\infty} a_n \neq 0$;
Since if $\lim\limits_{n\to\infty} a_n = 0$, $a_n < 1$ for $n \gg 1$ ✗ (by seq. loc'n thm.)
$\therefore \sum a_n$ diverges, by the n^{th} term test (Thm. 7.2)

6 Let $p_i = i^{th}$ prime, and $n_i = p_i^k$ (for a fixed k)
Then $\dfrac{h(n_i)}{s(n_i)} = \dfrac{p_i}{kp_i} = \dfrac{1}{k}$, so the subseq. $\dfrac{h(n_i)}{s(n_i)}$ converges to $1/k$, and $1/k$ is a cluster point of the sequence $\dfrac{h(n)}{s(n)}$, by the cluster pt. thm 6.2
Limit $\lim\limits_{n\to\infty}\dfrac{h(n)}{s(n)}$ doesn't exist, for if it had the limit L, all subsequences would have lim= (subseq. thm 5.4), which is not the case.

7 Choose x_n to be any element of S such that $x_n > n$.
Such an elt. exists since S is nonempty + not bounded above (if there were no such x_n, then n would be an upper bound for S).
Then $\lim\limits_{n\to\infty} x_n = \infty$, since (by Defn 3.3), given $M > 0$,
$x_n > n > M$ for all $n > M$, $\therefore$ for all $n \gg 1$.

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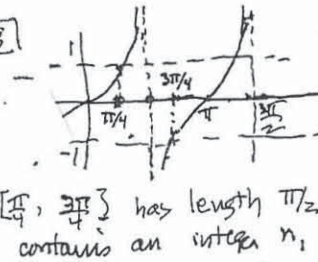

tan x has period π

$[\frac{\pi}{4}, \frac{3\pi}{4}]$ has length $\pi/2$, $\therefore$ contains an integer n_1
Similarly, for $[\pi/4 + k\pi, \frac{3\pi}{4} + k\pi]$ contains integer n_k.

$|\tan n_k| < 1$
$\therefore \tan n_k$ is bounded, and has a convergent subseq. $\tan n_{k_i}$ by B-W which is also a subseq. of $\tan n$.

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