18.100B : Fall 2010 : Section R2
Homework 10

Due Tuesday, November 16, 1pm

Reading: Tue Nov.9 : Riemann integral, Rudin 6.1-12 with $\alpha(x) = x$
Thu Nov.11 : holiday

1 . Consider the following two real-valued functions on $[0, 1]$.

$$f(x) = \begin{cases} 1, & x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}, \qquad g(x) = \begin{cases} n, & x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}.$$

Show (from the definition) that $f \in \mathscr{R}$ (i.e. f is Riemann-integrable), with $\int_0^1 f(x)\,dx = 0$, but $g \notin \mathscr{R}$.

2 . Problem 8, page 138 in Rudin.

3 . A subset $N \subseteq \mathbb{R}$ is said to have *measure* 0 if, for each $\epsilon > 0$, there is a sequence (finite or countable) of balls (B_n) with radii r_n so that $N \subseteq \bigcup_n B_n$ and $\sum_n r_n < \epsilon$.

(a) Let N be a set of measure 0 in $\mathbb{R}$. Prove that the complement N^c is dense in $\mathbb{R}$.

(b) Show (from the definition) that the only open set that has measure 0 is $\varnothing$.

(c) Can measure 0 sets be closed? Non-compact? Dense?

4 . Let $\mathscr{R}^1[a, b]$ denote the set of all functions $f\colon [a, b] \to \mathbb{R}$ with the property that $|f|$ is Riemann integrable. Define a function $d_1\colon \mathscr{R}^1[a, b] \times \mathscr{R}^1[a, b] \to \mathbb{R}_+$ by

$$d_1(f, g) = \int_a^b |f - g|.$$

(a) Show that if f is Riemann integrable on $[a, b]$, then $f \in \mathscr{R}^1[a, b]$.

(b) Is the converse of (a) true? [*Hint*: think about the function g that equals 1 on $\mathbb{Q}$ and 0 on $\mathbb{Q}^c$; find non-zero constants a, b so that $|ag + b|$ is constant.]

(c) Prove that d_1 satisfies all the axioms of a metric on $\mathscr{R}^1[a, b]$ except that $d_1(f, g) = 0$ for some $f \neq g$.

(d) Assume that f, g are continuous on $[a, b]$. Prove that $d_1(f, g) = 0$ iff $f = g$ on $[a, b]$. Conclude that the subset $\mathscr{C}[a, b]$ of continuous functions in $\mathscr{R}^1[a, b]$ is a metric space under the metric d_1.

5. Consider the metric space $(\mathscr{C}[-1,1], d_1)$ from **4(d)** (here $a=-1, b=1$).

(a) Let f_n denote the function

$$f_n(x) = \begin{cases} 1, & x > \frac{1}{n}, \\ nx, & 0 \le x \le \frac{1}{n} \\ 0, & x < 0 \end{cases}.$$

Show that $f_n \in \mathscr{C}[-1,1]$, and calculate $\int_0^1 f_n$.

(b) Show that the sequence (f_n) is a Cauchy sequence in terms of the metric d_1.

(c) Does f_n converge in $\mathscr{C}[-1,1]$? Is this metric space complete?