18.100B : Fall 2010 : Section R2
Homework 12

Due Tuesday, November 30, 1pm

Reading: Tue Nov.23 : sequences and series of functions, Rudin 7.1-17
Thu Nov.25 : holiday

1 . (a) Problem 2, page 165 in *Rudin*.

(b) Problem 3, page 165 in *Rudin*.

2 . Consider the exponential function $g(x) = e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$ on $\mathbb{R}$. Use Rudin Theorem 7.17 to prove that $g' = g$. (Hint: Note that an unguarded lecturer essentially did this in class, so full and glorious details are expected.)

3 . Problem 9, page 166 in *Rudin*.

4 . Problem 10, page 166 in *Rudin*. (Use some theorem from Rudin chapter 7 to prove Riemann-integrability.)

5 . Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function (i.e. all derivatives exist) and fix $x_0 \in \mathbb{R}$. The Taylor series $T(x)$ of f at x_0 is defined as the pointwise limit $T(x) = \lim_{k\to\infty} P_k(x)$ of the Taylor polynomials

$$P_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

(a) For which $r > 0$ do the Taylor polynomials P_k converge uniformly on $B_r(x_0)$ to T? (Hint: Write $T(x)$ as a series and compare r with its radius of convergence. You can use e.g. Rudin Theorem 7.10.)

(b) Recall Taylor's error formula for $|f(x) - P_k(x)|$. Deduce that $f = T$ on the ball $B_A(x_0)$ if $A > 0$ satisfies

$$A < \lim_{n\to\infty} \left(\frac{1}{n!} \sup_{z \in B_A(x_0)} |f^{(n)}(z)| \right)^{-1/n}.$$