18.100B Fall 2010
Practice Quiz 3 Solutions

1.(a) $R := f(x_0) - g(x_0) > 0$
f, g continuous, so with $\epsilon = R/2$ find
•$\exists \delta_f > 0 : f(B_{\delta_f}(x_0)) \subset (f(x_0) - R/2, f(x_0) + R/2)$
•$\exists \delta_g > 0 : g(B_{\delta_g}(x_0)) \subset (g(x_0) - R/2, g(x_0) + R/2)$
Take $r = \min\{\delta_f, \delta_g\} > 0$, then

$$y \in B_r(x_0) \Rightarrow \left\{ \begin{array}{l} f(y) > f(x_0) - R/2 \\ g(y) < g(x_0) + R/2 \end{array} \right\} \Rightarrow f(y) - g(y) > R - R/2 - R/2 = 0.$$

(b) To show: s continuous at any $x_0 \in X$
3 cases:
1) $f(x_0) > g(x_0)$: By (a), $s(x) = f(x)$ on $B_r(x_0)$ for some $r > 0$
f continuous $\Rightarrow s$ continuous at x_0
2) $f(x_0) < g(x_0)$: By (a), $s(x) = g(x)$ on $B_r(x_0)$ for some $r > 0$
g continuous $\Rightarrow s$ continuous at x_0
3) $f(x_0) = g(x_0)$: Given $\epsilon > 0$, find $\delta_f, \delta_g > 0$ and $\delta = \min\{\delta_f, \delta_g\} > 0$
$d(y, x_0) < \delta \Rightarrow s(B_\delta(x_0)) \subset (f(B_\delta(x_0)) \cup g(B_\delta(x_0))) \subset (s(x_0) - \epsilon, s(x_0) + \epsilon)$

(c) No

$$\left. \begin{array}{l} f(x) = x \\ g(x) = 0 \end{array} \right\} s(x) = \left\{ \begin{array}{ll} x; & x \le 0 \\ 0; & x > 0 \end{array} \right.$$

s is not differentiable at $x_0 = 0$ because

$$\left. \begin{array}{l} \displaystyle\lim_{t \to 0^+} \frac{s(t) - s(0)}{t - 0} = \lim_{t \to 0^+} \frac{0}{t} = \lim_{t \to 0^+} 0 = 0 \\ \displaystyle\lim_{t \to 0^-} \frac{s(t) - s(0)}{t - 0} = \lim_{t \to 0} \frac{t}{t} = \lim_{t \to 0} 1 = 1 \end{array} \right\} \Rightarrow \lim_{t \to 0} \frac{s(t) - s(0)}{t - 0} \text{ does not exist.}$$

2. Suppose by contradiction $f(U) = A \cup B$ is separated:
$\bar{A} \cap B = \emptyset, A \cap \bar{B} = \emptyset, A \neq \emptyset, B \neq \emptyset$
Then $U \subset f^{-1}(f(U))$ and hence
$U = f^{-1}(A \cup B) \cap U = (f^{-1}(A) \cup f^{-1}(B)) \cap U = A' \cup B'$
with $A' = f^{-1}(A) \cap U, B' = f^{-1}(B) \cap U$.
Claim: $U = A' \cup B'$ is separated in contradiction to assumption
Proof:
•$\exists a \in A \Rightarrow \exists a' \in U : f(a') = a \in A \Rightarrow \exists a' \in f^{-1}(A) \cap U = A' \Rightarrow A' \neq \emptyset$
• Similarly $B' \neq \emptyset$
• if $x \in \bar{A'} \cap B'$ then $x \in U, f(x) \in B$, and $\exists (x_n)_{n \in \mathbb{N}} \subset U : f(x_n) \in A, x_n \to x$

$$\left. \begin{array}{ll} \text{by continuity of } f, & f(x_n) \to f(x) \in B \\ \text{by } f(x_n) \in A, & f(x) = \lim f(x_n) \in \bar{A} \end{array} \right\} \text{contradiction to } \bar{A} \cap B = \emptyset$$

$\Rightarrow \bar{A'} \cap B' = \emptyset$
• Similarly $A' \cap \bar{B'} = \emptyset$
(or as in Rudin 4.22)

3.(a) $f : [a, b] \to \mathbb{R}$ continuous, differentiable on (a, b)
$\Rightarrow \exists x \in (a, b) : \dfrac{f(b) - f(a)}{b - a} = f'(x)$

(b)
Pick $0 < \epsilon < \frac{1}{M}$.
To check that f is injective, suppose by contradiction that $f(a) = f(b)$ for some $a < b$.
By the mean value theorem, there exists $x \in (a, b)$ s.t.
$0 = \dfrac{f(b) - f(a)}{b - a} = f'(x) = 1 + \epsilon g'(x) \geq 1 - \epsilon M > 0$
$\Rightarrow 0 > 0 \Rightarrow\Leftarrow$

(c)
By the mean value theorem,
$$\begin{matrix} \forall t > 0 & \exists x_t \in (0, t) : \\ \forall t < 0 & \exists x_t \in (t, 0) : \end{matrix} \; \frac{f(t) - f(0)}{t - 0} = f'(x_t)$$
Consider $t_n \to 0$, then $\dfrac{f(t_n) - f(0)}{t_n - 0} = f'(x_{t_n})$
where $0 < |x_{t_n}| < |t_n|$, hence $x_{t_n} \to 0$, and so by assumption ($\lim\limits_{x \to 0} f'(x) = A$),
$\dfrac{f(t_n) - f(0)}{t_n - 0} = f'(x_{t_n}) \to A$.
This shows $\lim\limits_{t \to 0} \dfrac{f(t) - f(0)}{t - 0} = A$, as claimed.

4.(a) TRUE
By the intermediate value theorem, $\forall R \in \mathbb{R} \; \exists a < 0, b > 0 : f(a) > R, f(b) < R$
$\Rightarrow \exists a < y < b : f(y) = R$

(b) TRUE
(Above $\Rightarrow$ uniformly continuous): Given ϵ, pick $N > \epsilon^{-1}$, which provides $R > 0$, then take $\delta \in (0, R^{-1})$
(Uniformly continuous $\Rightarrow$ above): Given N, let $\epsilon = 1/N$, which provides $\delta > 0$, then take $R > \delta^{-1}$
Uniformly continuous: $\forall \epsilon > 0 \; \exists \delta > 0 : \forall y f(B_\delta(y)) \subset B_\epsilon(f(y))$

(c) TRUE
$\lim\limits_{t \to 0} \dfrac{g(t) - g(0)}{t - 0} = \lim\limits_{t \to 0} \dfrac{tf(t)}{t} = \lim\limits_{t \to 0} f(t) = f(0)$ exists

(d) TRUE
$d(f(x), f(y)) = |f(x) - f(y)|$
$= |f'(z)(x - y)|$ for some $z \in \mathbb{R}$
$\leq Cd(x, y)$
So for $\epsilon > 0$ take $\delta = \dfrac{\epsilon}{C}$.

(e) FALSE

Idea: f can vary little with large f' by fast oscillation.

$f(x) = \dfrac{1}{x^2+1} \cdot \sin x^4$ is differentiable and uniformly continuous: Given $\epsilon > 0$

- Find $R > 0$ s.t. $\forall |x| > R, |f(x)| < \frac{\epsilon}{2}$
- Find $\delta > 0$ from continuity on $[-R, R]$

but $f'(x) = \dfrac{3x^3 \sin x^4 (x^2+1) - \sin x^4 \cdot 2x}{(x^2+1)^2}$

so for $x_n = \sqrt[4]{n\pi + \frac{\pi}{2}} \to \infty$ get $f'(x_n) \to \infty$