

18.100B Problem Set 10

Due Friday December 8, 2006 by 3 PM

Problems:

- 1) Let (f_n) be the sequence of functions on \mathbb{R} defined as follows.

$$f_0(t) = \sin t \quad \text{and} \quad f_{n+1}(t) = \frac{2}{3}f_n(t) + 1 \quad \text{for } n \in \mathbb{N}.$$

Show that $f_n \rightarrow 3$ uniformly on \mathbb{R} . What can you say if we choose $f_0(t) = t^2$?

Hint: Consider first the map $T(x) = \frac{2}{3}x + 1$ on \mathbb{R} .

- 2) Suppose $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is continuous and satisfies

$$0 \leq \varphi(t) \leq \frac{t}{2+t} \quad (t \geq 0).$$

Define the sequence (f_n) by setting $f_0(t) = \varphi(t)$ and $f_{n+1}(t) = \varphi(f_n(t))$ for $t \geq 0$ and $n \in \mathbb{N}$. Prove that the series $F(t) = \sum_{n=0}^{\infty} f_n(t)$ converges for every $t \geq 0$ and that F is continuous on $[0, \infty)$.

- 3) Does $f(t) = \sum_{k=1}^{\infty} \sin^2(t/k)$ define a differentiable function on \mathbb{R} ?
- 4) Suppose (f_n) is a sequence of continuous functions such that $f_n \rightarrow f$ uniformly on a set E . Prove that

$$\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \rightarrow x$, and $x \in E$. Is the converse of this true?

- 5) Suppose (f_n) is a sequence of real-valued functions that are Riemann-integrable on all compact subintervals of $[0, \infty)$. Assume further that:
- $f_n \rightarrow 0$ uniformly on every compact subset of $[0, \infty)$;
 - $0 \leq f_n(t) \leq e^{-t}$ for all $t \geq 0$ and $n \in \mathbb{N}$.

Prove that

$$\lim_{n \rightarrow \infty} \int_0^{\infty} f_n(t) dt = 0,$$

where the improper integral $\int_0^{\infty} f_n(t) dt$ is defined as $\lim_{b \rightarrow \infty} \int_0^b f_n(t) dt$. Moreover, give an explicit example for a sequence (f_n) , so that condition b) does not hold and the conclusion above fails.

Remark: In fact, one can relax condition a) to “ $f_n \rightarrow 0$ pointwise on $[0, \infty)$.” But then the proof (of this “dominated convergence theorem”) becomes by far more involved when using Riemann’s theory of integration.

- 6) Suppose (f_n) is an equicontinuous sequence of functions on a compact set K , and $f_n \rightarrow f$ pointwise on K . Prove that $f_n \rightarrow f$ uniformly on K .

- 7) Show that any uniformly bounded sequence of differentiable functions on a compact interval with uniformly bounded derivatives has a convergent subsequence.
(Hint: To apply the Arzela-Ascoli theorem (Thm 7.25) from the book, show that any family \mathcal{F} of real-valued, differentiable functions f defined on $[a, b]$, satisfying $|f'(x)| \leq M$ for some M and all $x \in [a, b]$ and $f \in \mathcal{F}$, must be equicontinuous.)

Extra problems:

- 1) In class, we have seen that uniform convergence of a sequence of bounded functions on a set E can be equivalently formulated in terms of the metric $d(f, g) = \sup_{x \in E} |f(x) - g(x)|$. That is, we have $d(f_n, f) \rightarrow 0$ if and only if $f_n \rightarrow f$ uniformly on E .

Having this in mind, we could ask whether an analogous statement holds with respect to pointwise convergence. More specifically, is there a metric $d(f, g)$ such that $d(f_n, f) \rightarrow 0$ if and only if $f_n \rightarrow f$ pointwise on E ? Surprisingly, it turns out that the answer is NO when, for example, $E = [0, 1]$. We therefore cordially invite you to prove the following theorem.

Theorem 1. *There is no metric $d(f, g)$ defined on $C([0, 1])$ such that $d(f_n, f) \rightarrow 0$ if and only if $f_n \rightarrow f$ pointwise on $[0, 1]$.*

Before proving this theorem, you may first show the following weaker statement whose proof requires less effort.

Theorem 2. *There is no norm $\|\cdot\|$ defined on $C([0, 1])$ such that $\|f_n\| \rightarrow 0$ if and only if $f_n \rightarrow 0$ pointwise on $[0, 1]$.*

Hint (for proof of Theorem 2). Consider $f_n \in C([0, 1])$, with $n = 1, 2, 3, \dots$, such that

- i) For every $x \in [0, 1]$, there exists $n_0 = n_0(x)$ such that $f_n(x) = 0$ if $n \geq n_0$.
- ii) $f_n \not\equiv 0$ for every $n \geq 1$.

(A possible choice is, for instance, given by $f_n(x) = \sin(n\pi x)$ if $x \in [0, 1/n]$, and $f_n(x) = 0$ if $x \in [1/n, 1]$.) By clever choice of a sequence of real-valued numbers (c_n) , prove the claim by considering the sequence $(c_n f_n)$.

Hint (for proof of Theorem 1). Assume there is such a metric $d(f, g)$ on $C([0, 1])$. Then $f_n \rightarrow 0$ if and only if, for every $k \in \mathbb{N}$, we have that $f_n \in N_{1/k}(0) = \{y \in C([0, 1]) : d(y, 0) < 1/k\}$, except for finitely many f_n . Use this fact and the specific choice $g_n(x) = e^{-n|x-x_0|}$ for suitable $x_0 \in [0, 1]$ to show that $g_n \rightarrow 0$ pointwise on $[0, 1]$, which is false! (Since $g(x_0) = 1$ for all n .)

- 2) Assume that (f_n) is a sequence of monotonically increasing functions on \mathbb{R} with $0 \leq f_n(x) \leq 1$ for all x and all n .
- (a) Prove that there is a function f and a sequence (n_k) such that

$$f(x) = \lim_{k \rightarrow \infty} f_{n_k}(x)$$

for every $x \in \mathbb{R}$. (This result is usually called *Helly's selection theorem*.)

- (b) If, moreover, f is continuous, prove that $f_{n_k} \rightarrow f$ uniformly on compact sets.

Hint: (i) Some subsequence (f_{n_i}) converges at all rational points r , say, to $f(r)$. (ii) Define $f(x) = \sup_{r \leq x} f(r)$ for any $x \in \mathbb{R}$. (iii) Show that $f_{n_i}(x) \rightarrow f(x)$ at every x at which f is continuous. (This is where monotonicity is strongly used.) (iv) A subsequence of (f_{n_i}) converges at every point of discontinuity of f , since there are at most countably many such points. This outlines the proof of (a). To prove (b), modify your proof of (iii) appropriately.

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