Daily Assignment for Lecture #36

Problem 1: Prove the Inverse Function Theorem for manifolds. A statement of the theorem and a proof of the sketch was given at the beginning of lecture #36, and is repeated at the bottom of this page.

Problem 2: Show that the unit n-sphere is oriented (Hint: read Munkres page 287).

Problem 3: Show that the linear map of $\mathbb{R}^{n+1}$ into itself mapping x to $-x$ is orientation preserving if and only if n is odd.

Problem 4: Show that the restriction of this map to the unit n-sphere is orientation preserving.

Problem 1 explained: Here we state the theorem and provide a sketch of the proof.

Let X, Y be n-dimensional manifolds, and let $f : X \to Y$ be a $\mathcal{C}^\infty$ map with $f(p) = p_1$.

Theorem. *If $df_p : T_pX \to T_{p_1}Y$ is bijective, then f maps a neighborhood V of p diffeomorphically onto a neighborhood V_1 of p_1.*

Sketch of proof: Let $\phi : U \to V$ be a parameterization of X at p, with $\phi(q) = p$. Similarly, let $\phi_1 : U_1 \to V_1$ be a parameterization of Y at p_1, with $\phi_1(q_1) = p_1$.

Show that we can assume that $f : V \to V_1$ (Hint: if not, replace V by $V \cap f^{-1}(V_1)$).

Show that we have a diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V_1 \\ {\scriptstyle \phi}\uparrow & & {\scriptstyle \phi_1}\uparrow \\ U & \xrightarrow{g} & U_1, \end{array} \tag{0.1}$$

which defines g,

$$g = \phi_1^{-1} \circ f \circ \phi, \tag{0.2}$$

$$g(q) = q_1. \tag{0.3}$$

So,

$$(dg)_q = (d\phi_1)_{q_1}^{-1} \circ df_p \circ (d\phi)_q. \tag{0.4}$$

Note that all three of the linear maps on the r.h.s. are bijective, so $(dg)_q$ is a bijection. Use the Inverse Function Theorem for open sets in $\mathbb{R}^n$. □