18.102 Introduction to Functional Analysis
Spring 2009

Lecture 19. Thursday, April 16

I am heading towards the spectral theory of self-adjoint compact operators. This is rather similar to the spectral theory of self-adjoint matrices and has many useful applications. There is a very effective spectral theory of general bounded but self-adjoint operators but I do not expect to have time to do this. There is also a pretty satisfactory spectral theory of non-selfadjoint compact operators, which it is more likely I will get to. There is no satisfactory spectral theory for general non-compact and non-self-adjoint operators as you can easily see from examples (such as the shift operator).

In some sense compact operators are 'small' and rather like finite rank operators. If you accept this, then you will want to say that an operator such as

$$\mathrm{Id} - K,\ K \in \mathcal{K}(\mathcal{H}) \tag{19.1}$$

is 'big'. We are quite interested in this operator because of spectral theory. To say that $\lambda \in \mathbb{C}$ is an eigenvalue of K is to say that there is a non-trivial solution of

$$Ku - \lambda u = 0 \tag{19.2}$$

where non-trivial means other than than the solution $u = 0$ which always exists. If $\lambda \neq 0$ we can divide by λ and we are looking for solutions of

$$(\mathrm{Id} - \lambda^{-1}K)u = 0 \tag{19.3}$$

which is just (19.1) for another compact operator, namely $\lambda^{-1}K$.

What are properties of $\mathrm{Id} - K$ which migh show it to be 'big? Here are three:

Proposition 26. *If $K \in \mathcal{K}(\mathcal{H})$ is a compact operator on a separable Hilbert space then*

$$\begin{gathered}\mathrm{null}(\mathrm{Id} - K) = \{u \in \mathcal{H}; (\mathrm{Id}_K)u = 0\} \text{ is finite dimensional}\\ \mathrm{Ran}(\mathrm{Id} - K) = \{v \in \mathcal{H}; \exists u \in \mathcal{H},\ v = (\mathrm{Id} - K)u\} \text{ is closed and}\\ \mathrm{Ran}(\mathrm{Id} - K)^{\perp} = \{w \in \mathcal{H}; (w, Ku) = 0\ \forall\ u \in \mathcal{H}\} \text{ is finite dimensional}\end{gathered} \tag{19.4}$$

and moreover

$$\dim\left(\mathrm{null}(\mathrm{Id} - K)\right) = \dim\left(\mathrm{Ran}(\mathrm{Id} - K)^{\perp}\right). \tag{19.5}$$

Definition 9. A bounded operator $F \in \mathcal{B}(\mathcal{H})$ on a Hilbert space is said to be *Fredholm* if it has the three properties in (19.4) – its null space is finite dimensional, its range is closed and the orthocomplement of its range is finite dimensional.

For general Fredholm operators the row-rank=colum-rank result (19.5) does not hold. Indeed the difference of these two integers

$$\mathrm{ind}(F) = \dim\left(\mathrm{null}(\mathrm{Id} - K)\right) - \dim\left(\mathrm{Ran}(\mathrm{Id} - K)^{\perp}\right) \tag{19.6}$$

is a very important number with lots of interesting properties and uses.

Notice that the last two conditions are really independent since the orthocomplement of a subspace is the same as the orthocomplement of its closure. There are for instance bounded opertors on a separable Hilbert space with trivial null space and dense range which is not closed. How could this be? Think for instance of the operator on $L^2(0,1)$ which is multiplication by the function x. This is assuredly bounded and an element of the null space would have to satisfy $xu(x) = 0$ almost everywhere, and hence vanish almost everywhere. Moreover the density of the L^2

functions vanishing in $x < \epsilon$ for some (non-fixed) $\epsilon > 0$ shows that the range is dense. However it is clearly not invertible.

Before proving this result let's check that the third condition in (19.4) really follows from the first. This is a general fact which I mentioned, at least, earlier but let me pause to prove it.

Proposition 27. *If $B \in \mathcal{B}(\mathcal{H})$ is a bounded operator on a Hilbert space and B^* is its adjoint then*

$$\operatorname{Ran}(B)^\perp = (\overline{\operatorname{Ran}}(B))^\perp = \{v \in \mathcal{H}; (v,w) = 0 \ \forall \ w \in \operatorname{Ran}(B)\} = \operatorname{Nul}(B^*). \tag{19.7}$$

Proof. The definition of the orthocomplement of $\operatorname{Ran}(B)$ shows immediately that

$$\begin{aligned} v \in (\operatorname{Ran}(B))^\perp \Longleftrightarrow (v,w) = 0 \ \forall \ w \in \operatorname{Ran}(B) \longleftrightarrow (v, Bu) = 0 \ \forall \ u \in \mathcal{H} \\ \Longleftrightarrow (B^*v, u) = 0 \ \forall \ u \in \mathcal{H} \Longleftrightarrow B^*v = 0 \Longleftrightarrow v \in \operatorname{Nul}(B^*). \end{aligned} \tag{19.8}$$

On the other hand we have already observed that $V^\perp = (\overline{B})^\perp$ for any subspace – since the right side is certainly contained in the left and $(u,v) = 0$ for all $v \in V$ implies that $(u,w) = 0$ for all $w \in \overline{V}$ by using the continuity of the inner product to pass to the limit of a sequence $v_n \to w$. □

Thus as a corrollary we see that if $\operatorname{Nul}(\operatorname{Id} - K)$ is always finite dimensional for K compact (i.e. we check it for all compact operators) then $\operatorname{Nul}(\operatorname{Id} - K^*)$ is finite dimensional and hence so is $\operatorname{Ran}(\operatorname{Id} - K)^\perp$.

Proof of Proposition 26. First let's check this in the case of a finite rank operator $K = T$. Then

$$\operatorname{Nul}(\operatorname{Id} - T) = \{u \in \mathcal{H}; u = Tu\} \subset \operatorname{Ran}(T). \tag{19.9}$$

A subspace of a finite dimensional space is certainly finite dimensional, so this proves the first condition in the finite rank case.

Similarly, still assuming that T is finite rank consider the range

$$\operatorname{Ran}(\operatorname{Id} - T) = \{v \in \mathcal{H}; v = (\operatorname{Id} - T)u \ \text{ for some } u \in \mathcal{H}\}. \tag{19.10}$$

Consider the subspace $\{u \in \mathcal{H}; Tu = 0\}$. We know that this this is closed, since T is certainly continuous. On the other hand from (19.10),

$$\operatorname{Ran}(\operatorname{Id} - T) \supset \operatorname{Nul}(T). \tag{19.11}$$

Remember that a finite rank operator can be written out as a finite sum

$$Tu = \sum_{i=1}^{N} (u, e_i) f_i \tag{19.12}$$

where we can take the f_i to be a basis of the range of T. We also know in this case that the e_i must be linearly independent – if they weren't then we could write one of them, say the last since we can renumber, out as a sum, $e_N = \sum\limits_{j<N} c_i e_j$, of multiples of the others and then find

$$Tu = \sum_{i=1}^{N-1} (u, e_i)(f_i + \overline{c_j} f_N) \tag{19.13}$$

showing that the range of T has dimension at most $N - 1$, contradicting the fact that the f_i span it.

So, going back to (19.12) we know that $\mathrm{Nul}(T)$ has finite *codimension* – every element of $\mathcal{H}$ is of the form

$$u = u' + \sum_{i=1}^{N} d_i e_i,\ u' \in \mathrm{Nul}(T). \tag{19.14}$$

So, going back to (19.11), if $\mathrm{Ran}(\mathrm{Id} - T) \neq \mathrm{Nul}(T)$, and it need not be equal, we can choose – using the fact that $\mathrm{Nul}(T)$ *is* closed – an element $g \in \mathrm{Ran}(\mathrm{Id} - T) \setminus \mathrm{Nul}(T)$ which is orthogonal to $\mathrm{Nul}(T)$. To do this, start with any a vector g' in $\mathrm{Ran}(\mathrm{Id} - T)$ which is not in $\mathrm{Nul}(T)$. It can be split as $g' = u'' + g$ where $g \perp \mathrm{Nul}(T)$ (being a closed subspace) and $u'' \in \mathrm{Nul}(T)$, then $g \neq 0$ is in $\mathrm{Ran}(\mathrm{Id} - T)$ and orthongonal to $\mathrm{Nul}(T)$. Now, the new space $\mathrm{Nul}(T) \oplus \mathbb{C}g$ is again closed and contained in $\mathrm{Ran}(\mathrm{Id} - T)$. But we can continue this process replacing $\mathrm{Nul}(T)$ by this larger closed subspace. After a a finite number of steps we conclude that $\mathrm{Ran}(\mathrm{Id} - T)$ itself is closed.

What we have just proved is:

Lemma 13. *If $V \subset \mathcal{H}$ is a subspace of a Hilbert space which contains a closed subspace of finite codimension in $\mathcal{H}$ – meaning $V \supset W$ where W is closed and there are finitely many elements $e_i \in \mathcal{H}$, $i = 1, \dots, N$ such that every element $u \in \mathcal{H}$ is of the form*

$$u = u' + \sum_{i=1}^{N} c_i e_i,\ c_i \in \mathbb{C}, \tag{19.15}$$

then V itself is closed.

So, this takes care of the case that $K = T$ has finite rank! What about the general case where K is compact? Here we just use a consequence of the approximation of compact operators by finite rank operators proved last time. Namely, if K is compact then there exists $B \in \mathcal{B}(\mathcal{H})$ and T of finite rank such that

$$K = B + T,\ \|B\| < \frac{1}{2}. \tag{19.16}$$

Now, consider the null space of $\mathrm{Id} - K$ and use (19.16) to write

$$\mathrm{Id} - K = (\mathrm{Id} - B) - T = (\mathrm{Id} - B)(\mathrm{Id} - T'),\ T' = (\mathrm{Id} - B)^{-1} T. \tag{19.17}$$

Here we have used the convergence of the Neumann series, so $(\mathrm{Id} - B)^{-1}$ does exist. Now, T' is of finite rank, by the ideal property, so

$$\mathrm{Nul}(\mathrm{Id} - K) = \mathrm{Nul}(\mathrm{Id} - T') \text{ is finite dimensional.} \tag{19.18}$$

Here of course we use the fact that $(\mathrm{Id} - K)u = 0$ is equivalent to $(\mathrm{Id} - T')u = 0$ since $\mathrm{Id} - B$ is invertible. So, this is the first condition in (19.4).

Similarly, to examine the second we do the same thing but the other way around and write

$$\mathrm{Id} - K = (\mathrm{Id} - B) - T = (\mathrm{Id} - T'')(\mathrm{Id} - B),\ T'' = T(\mathrm{Id} - B)^{-1}. \tag{19.19}$$

Now, T'' is again of finite rank and

$$\mathrm{Ran}(\mathrm{Id} - K) = \mathrm{Ran}(\mathrm{Id} - T'') \text{ is closed} \tag{19.20}$$

again using the fact that $\mathrm{Id} - B$ is invertible – so every element of the form $(\mathrm{Id} - K)u$ is of the form $(\mathrm{Id} - T'')u'$ where $u' = (\mathrm{Id} - B)u$ and conversely.

So, now we have proved all of (19.4) – the third part following from the first as discussed before.

What about (19.5)? This time let's first check that it is enough to consider the finite rank case. For a compact operator we have written

$$(\mathrm{Id} - K) = G(\mathrm{Id} - T) \tag{19.21}$$

where $G = \mathrm{Id} - B$ with $\|B\| < \frac{1}{2}$ is invertible and T is of finite rank. So what we want to see is that

$$\dim \mathrm{Nul}(\mathrm{Id} - K) = \dim \mathrm{Nul}(\mathrm{Id} - T) = \dim \mathrm{Nul}(\mathrm{Id} - K^*). \tag{19.22}$$

However, $\mathrm{Id} - K^* = (\mathrm{Id} - T^*)G^*$ and G^* is also invertible, so

$$\dim \mathrm{Nul}(\mathrm{Id} - K^*) = \dim \mathrm{Nul}(\mathrm{Id} - T^*) \tag{19.23}$$

and hence it is enough to check that $\dim \mathrm{Nul}(\mathrm{Id} - T) = \dim \mathrm{Nul}(\mathrm{Id} - T^*)$ – which is to say the same thing for finite rank operators.

Now, for a finite rank operator, written out as (19.12), we can look at the vector space W spanned by all the f_i's and all the e_i's together – note that there is nothing to stop there being dependence relations among the combination although separately they are independent. Now, $T : W \longrightarrow W$ as is immediately clear and

$$T^* v = \sum_{i=1}^{N} (v, f_i) e_i \tag{19.24}$$

so $T : W \longrightarrow W$ too. In fact $Tw' = 0$ and $T^* w' = 0$ if $w' \in W^\perp$ since then $(w', e_i) = 0$ and $(w', f_i) = 0$ for all i. It follows that if we write $R : W \longleftrightarrow W$ for the linear map on this finite dimensional space which is equal to $\mathrm{Id} - T$ acting on it, then R^* is given by $\mathrm{Id} - T^*$ acting on W and we use the Hilbert space structure on W induced as a subspace of $\mathcal{H}$. So, what we have just shown is that

$$(\mathrm{Id} - T)u = 0 \Longleftrightarrow u \in W \text{ and } Ru = 0,\ (\mathrm{Id} - T^*)u = 0 \Longleftrightarrow u \in W \text{ and } R^* u = 0. \tag{19.25}$$

Thus we really are reduced to the finite-dimensional theorem

$$\dim \mathrm{Nul}(R) = \dim \mathrm{Nul}(R^*) \text{ on } W. \tag{19.26}$$

You no doubt know this result. It follows by observing that in this case, everything now on W, $\mathrm{Ran}(W) = \mathrm{Nul}(R^*)^\perp$ and finite dimensions

$$\dim \mathrm{Nul}(R) + \dim \mathrm{Ran}(R) = \dim W = \dim \mathrm{Ran}(W) + \dim \mathrm{Nul}(R^*). \tag{19.27}$$

□

Problem set 9, Due 11AM Tuesday 28 Apr.

My apologies for all these errors. Here is a list – they are fixed below (I hope).

(1) In P9.2 (2), and elsewhere, $\mathcal{C}^\infty(\mathbb{S})$ should be $\mathcal{C}^0(\mathbb{S})$, the space of continuous functions on the circle – with supremum norm.
(2) In (19.40) it should be $u = Fv$, not $u = Sv$.
(3) Similarly, before (19.41) it should be $u = Fv$.
(4) Discussion around (19.43) clarified.
(5) Last part of P10.2 clarified.

This week I want you to go through the invertibility theory for the operator

$$Qu = (-\frac{d^2}{dx^2} + V(x))u(x) \tag{19.28}$$

acting on periodic functions. Since we have not developed the theory to handle this directly we need to approach it through integral operators.

Before beginning, we need to consider periodic functions.

P9.1: Periodic functions

Let $\mathbb{S}$ be the circle of radius 1 in the complex plane, centered at the origin, $\mathbb{S} = \{z; |z| = 1\}$.

(1) Show that there is a 1-1 correspondence

$$\begin{aligned}\mathcal{C}^0(\mathbb{S}) = \{u : \mathbb{S} \longrightarrow \mathbb{C},\ \text{continuous}\} \longrightarrow \\ \{u : \mathbb{R} \longrightarrow \mathbb{C};\ \text{continuous and satisfying } u(x+2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}.\end{aligned} \tag{19.29}$$

(2) Show that there is a 1-1 correspondence

$$\begin{aligned}L^2(0,2\pi) \longleftrightarrow \{u \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}); u\big|_{(0,2\pi)} \in \mathcal{L}^2(0,2\pi) \\ \text{and } u(x+2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}/\mathcal{N}_P\end{aligned} \tag{19.30}$$

where $\mathcal{N}_P$ is the space of null functions on $\mathbb{R}$ satisfying $u(x+2\pi) = u(x)$ for all $x \in \mathbb{R}$.

(3) If we denote by $L^2(\mathbb{S})$ the space on the left in (19.30) show that there is a dense inclusion

$$\mathcal{C}^0(\mathbb{S}) \longrightarrow L^2(\mathbb{S}). \tag{19.31}$$

So, the idea is that we can think of functions on $\mathbb{S}$ as 2π-periodic functions on $\mathbb{R}$.

P9.2: Schrödinger's operator

Since that is what it is, or at least it is an example thereof:

$$-\frac{d^2u(x)}{dx^2} + V(x)u(x) = f(x),\ x \in \mathbb{R}, \tag{19.32}$$

(1) First we will consider the special case $V = 1$. Why not $V = 0$? – Don't try to answer this until the end!
(2) Recall how to solve the differential equation

$$-\frac{d^2u(x)}{dx^2} + u(x) = f(x),\ x \in \mathbb{R}, \tag{19.33}$$

where $f(x) \in \mathcal{C}^0(\mathbb{S})$ is a continuous, 2π-periodic function on the line. Show that there is a unique 2π-periodic and twice continuously differentiable function, u, on $\mathbb{R}$ satisfying (19.33) and that this solution can be written in the form

$$u(x) = (Sf)(x) = \int_{0,2\pi} A(x,y)f(y) \tag{19.34}$$

where $A(x,y) \in \mathcal{C}^0(\mathbb{R}^2)$ satisfies $A(x+2\pi, y+2\pi) = A(x,y)$ for all $(x,y) \in \mathbb{R}$.

Extended hint: In case you managed to avoid a course on differential equations! First try to find a solution, igonoring the periodicity issue. To do so one can (for example, there are other ways) factorize the differential operator involved, checking that

$$-\frac{d^2u(x)}{dx^2} + u(x) = -(\frac{dv}{dx} + v) \text{ if } v = \frac{du}{dx} - u \tag{19.35}$$

since the cross terms cancel. Then recall the idea of integrating factors to see that

$$\begin{aligned} \frac{du}{dx} - u &= e^x \frac{d\phi}{dx},\ \phi = e^{-x}u, \\ \frac{dv}{dx} + v &= e^{-x}\frac{d\psi}{dx},\ \psi = e^x v. \end{aligned} \tag{19.36}$$

Now, solve the problem by integrating twice from the origin (say) and hence get a solution to the differential equation (19.33). Write this out explicitly as a double integral, and then change the order of integration to write the solution as

$$u'(x) = \int_{0,2\pi} A'(x,y)f(y)dy \tag{19.37}$$

where A' is continuous on $\mathbb{R}\times[0,2\pi]$. Compute the difference $u'(2\pi) - u'(0)$ and $\frac{du'}{dx}(2\pi) - \frac{du'}{dx}(0)$ as integrals involving f. Now, add to u' as solution to the *homogeneous* equation, for $f = 0$, namely $c_1e^x + c_2e^{-x}$, so that the new solution to (19.33) satisfies $u(2\pi) = u(0)$ and $\frac{du}{dx}(2\pi) = \frac{du}{dx}(0)$. Now, check that u is given by an integral of the form (19.34) with A as stated.

(3) Check, either directly or indirectly, that $A(y,x) = A(x,y)$ and that A is real.
(4) Conclude that the operator S extends by continuity to a bounded operator on $L^2(\mathbb{S})$.
(5) Check, probably indirectly rather than directly, that

$$S(e^{ikx}) = (k^2+1)^{-1}e^{ikx},\ k \in \mathbb{Z}. \tag{19.38}$$

(6) Conclude, either from the previous result or otherwise that S is a compact self-adjoint operator on $L^2(\mathbb{S})$.
(7) Show that if $g \in \mathcal{C}^0(\mathbb{S}))$ then Sg is twice continuously differentiable. Hint: Proceed directly by differentiating the integral.
(8) From (19.38) conclude that $S = F^2$ where F is also a compact self-adjoint operator on $L^2(\mathbb{S})$ with eigenvalues $(k^2+1)^{-\frac{1}{2}}$.
(9) Show that $F : L^2(\mathbb{S}) \longrightarrow \mathcal{C}^0(\mathbb{S})$.
(10) Now, going back to the real equation (19.32), we assume that V is continuous, real-valued and 2π-periodic. Show that if u is a twice-differentiable

2π-periodic function satisfying (19.32) for a given $f \in \mathcal{C}^0(\mathbb{S})$ then

$$u + S((V-1)u) = Sf \text{ and hence } u = -F^2((V-1)u) + F^2 f \tag{19.39}$$

and hence conclude that

$$u = Fv \text{ where } v \in L^2(\mathbb{S}) \text{ satisfies } v + (F(V-1)F)v = Ff \tag{19.40}$$

where $V-1$ is the operator defined by multiplication by $V-1$.

(11) Show the converse, that if $v \in L^2(\mathbb{S})$ satisfies

$$v + (F(V-1)F)v = Ff,\ f \in \mathcal{C}^0(\mathbb{S}) \tag{19.41}$$

then $u = Fv$ is 2π-periodic and twice-differentiable on $\mathbb{R}$ and satisfies (19.32).

(12) Apply the Spectral theorem to $F(V-1)F$ (including why it applies) and show that there is a sequence λ_j in $\mathbb{R} \setminus \{0\}$ with $|\lambda_j| \to 0$ such that for all $\lambda \in \mathbb{C} \setminus \{0\}$, the equation

$$\lambda v + (F(V-1)F)v = g,\ g \in L^2(\mathbb{S}) \tag{19.42}$$

has a unique solution for every $g \in L^2(\mathbb{S})$ if and only if $\lambda \neq \lambda_j$ for any j.

(13) Show that for the λ_j the solutions of

$$\lambda_j v + (F(V-1)F)v = 0, v \in L^2(\mathbb{S}), \tag{19.43}$$

are all continuous 2π-periodic functions on $\mathbb{R}$.

(14) Show that the corresponding functions $u = Fv$ where v satisfies (19.43) are all twice continuously differentiable, 2π-periodic functions on $\mathbb{R}$ satisfying

$$-\frac{d^2u}{dx^2} + (1 - s_j + s_jV(x))u(x) = 0,\ s_j = 1/\lambda_j. \tag{19.44}$$

(15) Conversely, show that if u is a twice continuously differentiable, 2π-periodic function satisfying

$$-\frac{d^2u}{dx^2} + (1 - s + sV(x))u(x) = 0,\ s \in \mathbb{C}, \tag{19.45}$$

and u is not identically 0 then $s = s_j$ for some j.

(16) Finally, conclude that Fredholm's alternative holds for the equation (19.32)

Theorem 14. *For a given real-valued, continuous 2π-periodic function V on $\mathbb{R}$, either* (19.32) *has a unique twice continuously differentiable, 2π-periodic, solution for each f which is continuous and 2π-periodic or else there exists a finite, but positive, dimensional space of twice continuously differentiable 2π-periodic solutions to the homogeneous equation*

$$-\frac{d^2w(x)}{dx^2} + V(x)w(x) = 0,\ x \in \mathbb{R}, \tag{19.46}$$

and (19.32) *has a solution if and only if $\int_{(0,2\pi)} fw = 0$ for every 2π-periodic solution, w, to* (19.46).

Not to be handed in, just for the enthusiastic

Check that we really can understand all the 2π periodic eigenfunctions of the Schrödinger operator using the discussion above. First of all, there was nothing sacred about the addition of 1 to $-d^2/dx^2$, we could add any positive number and get a similar result – the problem with 0 is that the constants satisfy the homogeneous equation $d^2u/dx^2 = 0$. What we have shown is that the operator

$$u \longmapsto Qu = -\frac{d^2u}{dx^2}u + Vu \tag{19.47}$$

applied to twice continuously differentiable functions has at least a left inverse unless there is a non-trivial solution of

$$-\frac{d^2u}{dx^2}u + Vu = 0. \tag{19.48}$$

Namely, the left inverse is $R = F(\mathrm{Id} + F(V-1)F)^{-1}F$. This is a compact self-adjoint operator. Show – and there is still a bit of work to do – that (twice continuously differentiable) eigenfunctions of Q, meaning solutions of $Qu = \tau u$ are precisely the non-trivial solutions of $Ru = \tau^{-1}u$.

What to do in case (19.48) does have a non-trivial solution? Show that the space of these is finite dimensional and conclude that essentially the same result holds by working on the orthocomplement in $L^2(\mathbb{S})$.