Problem Set 3

AG §1.4, pp 49–52: 10, 19. (Problem 10 elaborates on the meaning of independence. Problem 19 shows that there is no model of Bernoulli trials in a countable probability space.)

AG §2.1, pp 58–60: 2, 3.

AG §2.2, pp 69–72: 1, 3, 5, 9

(Problem *) Existence of a set of real numbers that is not Lebesgue measurable

Notations. Let $\mathbf{R}$ denote the real numbers and $\mathbf{Q}$ the rational numbers. As in §1.3/14, if $E \subset \mathbf{R}$, denote

$$E + c = \{x + c : x \in E\}$$

Let $I = [0, 1)$, the half-open interval.

a) Show that there exists a set $E \subset I$ such that for every $x \in \mathbf{R}$ there exists a unique $x' \in E$ such that $x - x' \in \mathbf{Q}$. (This step uses the axiom of choice.)

b) Show that if q_1 and q_2 are distinct rational numbers, then $(E + q_1) \cap (E + q_2) = \emptyset$.

c) Show that

$$[0, 1) \subset \bigcup_{q \in \mathbf{Q}, |q| \leq 1} E + q \subset [-1, 2),$$

d) Deduce that E is not Lebesgue measurable.

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