Twin Prime Conjecture

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Introduction

“Twin Prime” – Paul Stackel, 1880s

{p, p+2}   equivalently, {6n+1, 6n-1}:

$$6x + 0 \neq \text{prime} = 6x$$

$$6x + 1 = \text{prime}$$

$$6x + 2 \neq \text{prime} = 2(3x+1)$$

$$6x + 3 \neq \text{prime} = 3(2x+1)$$

$$6x + 4 \neq \text{prime} = 2(3x+2)$$

$$6x + 5 = \text{prime} = 6y - 1$$

The Prime Counting Function and the Twin Prime Constant

Twin Prime Counting Function:

Prime Counting Function

$$\pi(x) = \{N(p) | p \leqq x\}$$

$$\pi_2(x) \leqq c\Pi_2 \frac{x}{(\ln(x))^2}[1 + O(\frac{\ln(\ln(x))}{\ln(x)})]$$

Formulated by Mertens

Twin Prime Constant

$$\prod_2 \quad := \quad \prod(1 - \frac{1}{9p-1)^2})$$

$$\pi_2(x) \sim 2\Pi_2 \int_2^x \frac{dx}{(\ln(x))^2}.$$

Formulated by Hardy and Littlewood

Mertens' Theorems

Mertens Theorem 1:

For any real number $x \geq 1$,

$$0 \leq \sum_{n \leq x} \ln(\frac{x}{n}) < x.$$

The function $f(t) = \ln(\frac{x}{t})$ is decreasing on the interval $[1, x]$, so

$$\sum_{1 \leq n \leq x} \ln(\frac{x}{n}) < \ln(x) + \int_1^x \ln(\frac{x}{t})dt$$

We can rewrite the right-hand side of the inequality as the following:

$$\ln(x) + \int_1^x \ln(\frac{x}{t})dt = x\ln(x) - \int_1^x \ln(t)dt.$$

Similarly, we can rewrite this:

$$x\ln(x) - \int_1^x \ln(t)dt = x\ln(x) - (x\ln(x) - x + 1) < x.$$

Mertens' Second Theorem

For Mertens' second theorem, we introduce the Von Mangoldt's function, $\Lambda(n)$, where

$$\Lambda(n) = \ln(p) \text{ if } n = p^m \text{ is a prime power, and zero otherwise.}$$

Then the psi function of the prime number theorem is defined as follows

$$\Psi(x) = \sum_{1 \le m \le x} \Lambda(m).$$

Mertens Theorem 2:
For any real number $x \geq 1$,

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \ln(x) + O(1).$$

Proof:
Let $N = [x]$. Then

$$0 \leq \sum_{n \leq x} \ln(\frac{x}{n}) = N \ln(x) - \sum_{n=1}^{N} \ln(n) = x \ln(x) - \ln(N!) + O(\ln(x)) < x$$

$$\ln(N!) = x \ln(x) + O(x).$$

Let $v_p(n)$ denote the highest power of p, a prime, that divides n.

$$\ln(N!) = \sum_{p \le N} v_p(N) \ln(p)$$

We can rewrite this as a single summation, by combining the limits on p and k:

$$\ln(N!) = \sum_{p \le N} \sum_{k=1}^{[\frac{\ln(N)}{\ln(p)}]} [\frac{N}{p^k}] \ln(p).$$

$$\ln(x!) = \sum_{n \le x} [\frac{x}{n}] \Lambda(n).$$

$$\ln(x!) = \sum_{n \le x} (\frac{x}{n} + O(1)) \Lambda(n).$$

$$\ln(x!) = \sum_{n \leq x} (\frac{x}{n} + O(1))\Lambda(n).$$

We can distribute this term, forming two sums, one in the error term:

$$\ln(x!) = x \sum_{n \leq x} \frac{\Lambda(n)}{n} + O(\sum_{n \leq x} \Lambda(n)).$$

Now we can substitute in the Psi function defined earlier:

$$\ln(x!) = x \sum_{n \leq x} \frac{\Lambda(n)}{n} + O(\Psi(x)).$$

Since the Psi function is of the same order as a linear function in x, we can replace it in the error term, obtaining the following:

$$\ln(x!) = x \sum_{n \leq x} \frac{\Lambda(n)}{n} + O(x).$$

Therefore,

$$x \sum_{n \leq x} \frac{\Lambda(n)}{n} + O(x) = x \ln(x) + O(x)$$

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \ln(x) + O(1)$$

Mertens Theorem 3:

For any real number $x \geq 1$,

$$\sum_{p \leq x} \frac{\ln(p)}{p} = \ln(x) + O(1).$$

Proof:

From the previous theorem,

$$0 \leq \sum_{n \leq x} \frac{\Lambda(n)}{n} - \sum_{p \leq x} \frac{\ln(p)}{p} = \sum_{p^k \leq x, k \geq 2} \frac{\ln(p)}{p^k}$$

$$\leq \sum_{p \leq x} \ln(p) \sum_{k=2}^{\infty} \frac{a}{p^k} \leq \sum_{p \leq x} \frac{\ln(p)}{p(p-1)}$$

$$\leq 2 \sum_{p \leq x} \frac{\ln(p)}{p^2} \leq 2 \sum_{n=1}^{\infty} \frac{\ln(n)}{n^2} = O(1)$$

It then follows from the previous theorem that

$$\sum_{p \le x} \frac{\ln(p)}{p} = \sum_{n \le x} \frac{\Lambda(n)}{n} + O(1) = \ln(x) + O(1).$$

Mertens' Theorem 4:

There exists a constant $b_1 > 0$ such that

$$\sum_{p \le x} \frac{1}{p} = \ln(\ln(x)) + b_1 + O(\frac{1}{\ln(x)}), x \ge 2.$$

This shows that the sum of reciprocals of primes diverge, whereas the reciprocals of twin primes converge

We can write

$$\sum_{p \le x} \frac{1}{p} = \sum_{p \le x} \frac{\ln(p)}{p} \frac{1}{\ln(p)} = \sum_{n \le x} u(n) f(n)$$

where $u(n) = \frac{\ln(p)}{p}$ if $n = p$, and 0 otherwise, and $f(t) = \frac{1}{\ln(t)}$.

Let $$U(t) = \sum_{n \le t} u(n) = \sum_{p \le t} \frac{\ln(p)}{p} = \ln(t) + g(t)$$

Then U(t) = 0 for t < 2 and g(t) = O(1) by our assumption

$$\int_x^\infty \frac{g(t)dt}{t(\ln(t))^2} = O(\frac{1}{\ln(x)}).$$

$$\sum_{p\le x}\frac{1}{p}=\sum_{n\le x}u(n)f(n)=\frac{1}{2}+\int_2^x f(t)dU(t)$$

Integrating by parts, we obtain the following:

$$\frac{1}{2}+\int_2^x f(t)dU(t)=f(x)U(x)-\int_2^x U(t)df(t)=\frac{\ln(x)+g(x)}{\ln(x)}-\int_2^x U(t)f'(t)dt.$$

Now we can simplify the term outside the integral, and substitute in for $U(t)$:

$$\frac{1}{2}+\int_2^x f(t)dU(t)=1+O(\frac{1}{\ln(x)})+\int_2^x \frac{\ln(t)+g(t)}{t(\ln(t))^2}dt.$$

We can split the integral in order to simplify the result:

$$\frac{1}{2}+\int_2^x f(t)dU(t)=\int_2^x \frac{1}{t\ln(t)}dt+\int_2^\infty \frac{g(t)}{t(\ln(t))^2}dt-\int_x^\infty \frac{g(t)}{t(\ln(t))^2}dt+1+O(\frac{1}{\ln(x)}).$$

Now we can evaluate two of the integrals:

$$\int_2^x \frac{1}{t\ln(t)}dt + \int_2^\infty \frac{g(t)}{t(\ln(t))^2}dt = \ln(\ln(x)) - \ln(\ln(2))$$

Finally, we can simplify this result in terms of a variable b_1:

$$\ln(\ln(x)) - \ln(\ln(2)) + \int_2^\infty \frac{g(t)}{t(\ln(t))^2}dt + 1 + O(\frac{1}{\ln(x)}) = \ln(\ln(x)) + b_1 + O(\frac{1}{\ln(x)})$$

where

$$b_1 = 1 - \ln(\ln(2)) + \int_2^\infty \frac{g(t)}{t(\ln(t))^2}dt.$$

Now we not only know that the reciprocals of primes diverge, but that they diverge like the function ln(ln(x)).

Brun's Conjecture:

Let p_1, p_2, ... be the sequence of prime numbers p such that $p+2$ is also prime. Then

$$\sum_{n=1}^{\infty}\left(\frac{1}{p_n}+\frac{1}{p_n+2}\right)=\left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\ldots<\infty$$

$\pi_2(x) << \frac{x}{(\ln(x))^{\frac{3}{2}}}$ for all $x \geq 2$.

$$n=\pi_2(p_n)<\frac{p_n}{(\ln(p_n))^{\frac{3}{2}}}\leq\frac{p_n}{(\ln(n))^{\frac{3}{2}}}$$

for $n \geq 2$. Then

$$\frac{1}{p_n}<\frac{1}{n(\ln(n))^{\frac{3}{2}}}.$$

It follows that the series defined above is convergent:

$$\sum_{n=1}^{\infty}\frac{1}{p_n}\leq\frac{1}{3}+\sum_{n=2}^{\infty}\frac{1}{p_n}<<\frac{1}{3}+\sum_{n=2}^{\infty}\frac{1}{n(\ln(n))^{\frac{3}{2}}}.$$

$$\pi_2(x) << \frac{x}{(\ln(x))^{\frac{3}{2}}} \text{ for all } x \geq 2.$$

(This result, which we assumed in the last theorem, actually has an involved proof using the Brun Sieve technique)

Fun Exercise: How many primes are in an interval?

We can first evaluate this by using Euler's expression for the prime counting function.

$$\pi(x+\epsilon x) - \pi(x) = \frac{x+\epsilon x}{\ln(x)+\ln(1+\epsilon)} - \frac{x}{\ln(x)} + O(\frac{x}{\ln(x)}).$$

We can rewrite the right-hand side as

$$\frac{\epsilon x}{\ln(x)} + O(\frac{x}{\ln(x)})$$

Then if we let $\epsilon = 1$,

$$\pi(2x) - \pi(x) = \frac{x}{\ln(x)} + O(\frac{x}{\ln(x)})\ \pi(x)$$

This does not mean that the number of primes in an interval of length n is equal to the number of primes in the sequential interval of length n. Instead, it means that

$$\pi(2x) - 2\pi(x) = O(\pi(x))$$

Conclusion

The infinitude of twin primes has not been proven, but current work by Dan Goldston and Cem Yilidrim is focused on a formula for the interval between two primes:

$$\Delta = \liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\ln(p_n)} = 1$$