The Sine Product Formula and the Gamma Function

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Outline

- Introduction
- Weierstrass' Product Formula
- Multiplication Formula
- Sine and Gamma Functions
- Applications of Sine Product Formula

Introduction

Useful Formulas:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$$

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(n) = (n-1)!$$

Introduction

More Useful Formulas:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Weierstrass' Product Formula

Theorem (Gauss):

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\cdots(x+n)}$$

Weierstrass' Product Formula

Weierstrass' Product Formula

$$\Gamma(x) = e^{-Cx}\frac{1}{x}\prod_{i=1}^{\infty}\frac{e^{x/i}}{1+x/i},$$

$$\text{where } C = \lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right)$$

Multiplication Formula

Gauss' Multiplication Formula

$$\frac{(2\pi)^{(p-1)/2}}{p^{x-1/2}}\Gamma(x)=\Gamma\left(\frac{x}{p}\right)\Gamma\left(\frac{x+1}{p}\right)\cdots\Gamma\left(\frac{x+p-1}{p}\right)$$

Legendre's Relation, where p=2

$$\frac{\sqrt{\pi}}{2^{x-1}}\Gamma(x)=\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right)$$

Sine and Gamma Functions

Define:

$$\phi(x) = \Gamma(x)\Gamma(1-x)\sin \pi x,$$

$$\text{then } \phi(x+1) = \phi(x).$$

Sine and Gamma Functions

Proof:

$$\Gamma(-x+1) = -x\Gamma(-x)$$

$$\Gamma(-x) = \frac{\Gamma(-x+1)}{-x}$$

$$\phi(x+1) = \Gamma(x+1)\Gamma(-x)\sin(\pi(x+1))$$

$$\phi(x+1) = x\Gamma(x)\frac{\Gamma(-x+1)}{-x}(-\sin \pi x)$$

$$\phi(x+1) = \Gamma(x)\Gamma(-x+1)\sin \pi x = \phi(x)$$

Sine and Gamma Functions

$$b2^{-x}\Gamma(x)=\Gamma(\frac{x}{2})\Gamma(\frac{x+1}{2})$$

$$b2^{x-1}\Gamma(1-x)=\Gamma(\frac{1-x}{2})\Gamma(1-\frac{x}{2})$$

Sine and Gamma Functions

$$\phi(\frac{x}{2})\phi(\frac{x+1}{2})$$

$$= \Gamma(\frac{x}{2})\Gamma(1-\frac{x}{2})\sin\frac{\pi x}{2}\Gamma(\frac{x+1}{2})\Gamma(\frac{1-x}{2})\cos\frac{\pi x}{2}$$

$$= \frac{b^2}{4}\Gamma(x)\Gamma(1-x)\sin\pi x$$

$$= \frac{b^2}{4}\phi(x)$$

Sine and Gamma Functions

$$\phi(x) = \Gamma(x)\Gamma(1-x)\sin \pi x$$

$$= \frac{\Gamma(1+x)}{x}\Gamma(1-x)(\pi x - \frac{\pi x^3}{3!} + \frac{\pi x^5}{5!} - \frac{\pi x^7}{7!} + \cdots)$$

$$= \Gamma(1+x)\Gamma(1-x)(\pi - \frac{\pi^3 x^2}{3!} + \frac{\pi^5 x^4}{5!} - \frac{\pi^7 x^6}{7!} + \cdots)$$

$$\phi(0) = \pi$$

Sine and Gamma Functions

- Define g(x) to be a periodic function, which is the second derivative of log(phi(x)).
- It is bounded and the bound of g(x) goes to 0, so g(x)=0 and log(phi(x)) is linear.
- Since log(phi(x)) is periodic, it must be constant.
- Therefore phi(x) is constant and equals pi for all x.

Sine and Gamma Functions

Since $\phi(x) = \pi$,

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

$$\sin \pi x = \frac{\pi}{-x\Gamma(x)\Gamma(-x)}$$

Sine and Gamma Functions

Sine Product Formula

$$\sin \pi x = \pi x \prod_{i=1}^{\infty} (1 - \frac{x^2}{i^2})$$

Applications

Riemann Zeta Function:

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots$$

$$\zeta(2)=\frac{\pi^2}{6}$$

$$\zeta(4)=\frac{\pi^4}{90}$$

Applications

From zeta(2):

$$\sum_{i=1}^{\infty}\frac{1}{i^2\pi^2}=\frac{1}{6}$$

$$\left(\sum_{i=1}^{\infty}\frac{1}{i^2\pi^2}\right)^2=\sum_{i=1}^{\infty}\frac{1}{i^4\pi^4}+2\sum_{i\neq j}\frac{1}{i^2j^2\pi^4}=\frac{1}{36}$$

Applications

Using the Sine Product Formula:

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots$$

The coefficient of x^4:

$$\sum_{i \neq j} \frac{1}{i^2 j^2 \pi^4} = \frac{1}{5!} = \frac{1}{120}$$

Applications

$$\left(\sum_{i=1}^{\infty}\frac{1}{i^2\pi^2}\right)^2=\sum_{i=1}^{\infty}\frac{1}{i^4\pi^4}+2\sum_{i\neq j}\frac{1}{i^2j^2\pi^4}$$

$$\frac{1}{36}=\sum_{i=1}^{\infty}\frac{1}{i^4\pi^4}+2\left(\frac{1}{120}\right)$$

$$\zeta(4)=\sum_{i=1}^{\infty}\frac{1}{i^4}=\frac{\pi^4}{90}$$