The Golden Ratio

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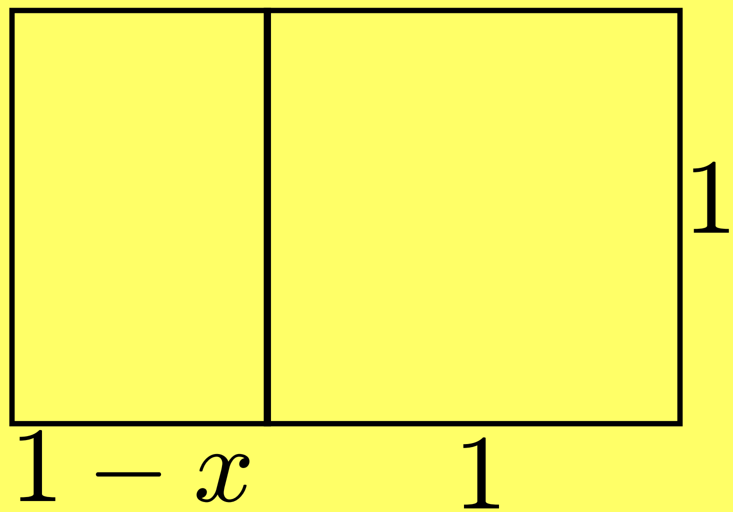
Outline

- Geometric Definition
- Relation with Fibonacci Numbers
- Euclidean Geometric Construction
- Continuous Fraction Representation

Geometric Definition (Mean and Extreme Ratio)



- x satisfies

$$x^2 - x - 1 = 0$$

- Golden ratio = positive root = $\tau = \frac{1+\sqrt{5}}{2}$
- Negative root = 1 – $\tau = \mu = \frac{1-\sqrt{5}}{2}$

Relation with Fibonacci Numbers

- Binet's Formula

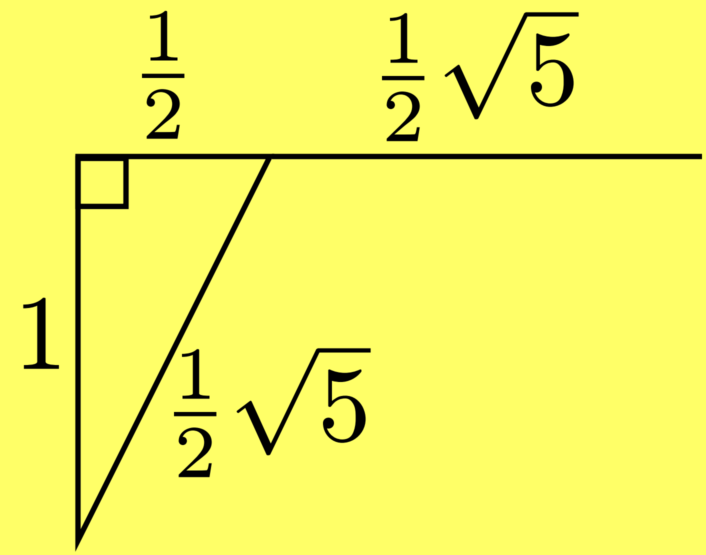
$$f_n = (\tau^n - \mu^n)/\sqrt{5}$$

- $$\frac{f_{n+1}}{f_n} = \frac{\tau^{n+1} - \mu^{n+1}}{\tau^n - \mu^n}$$

- $\tau = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}$ since $|\tau/\mu| > 1$

Geometric Construction

- Construct a right triangle with sides $\frac{1}{2}$ and 1
- Add the hypotenuse and shortest side



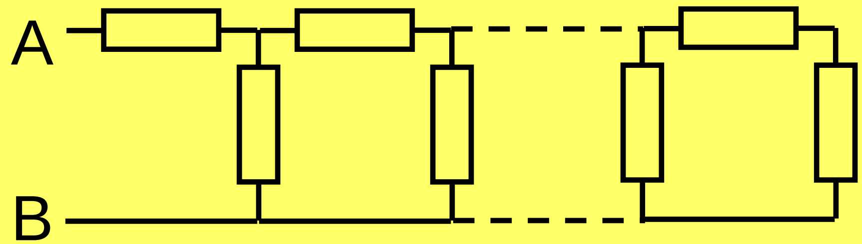
Continuous Fraction Representation

$$\tau = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

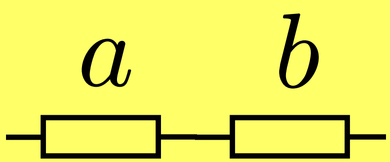
- $\tau = \lim_{n\to\infty} u_n$, where $u_{n+1} = 1 + \frac{1}{u_n}$ and $u_1 = 1$
- Let $u_n = a_{n+1}/a_n$ with $a_1 = a_2 = 1$
- Recursion of a_n is $a_{n+2} = a_{n+1} + a_n$
- $\{a_n\} =$ Fibonacci numbers

Infinite Resistor Network

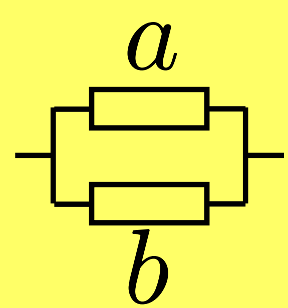
- Each resistor has resistance 1Ω



- Total resistance = r = ?
- Recall

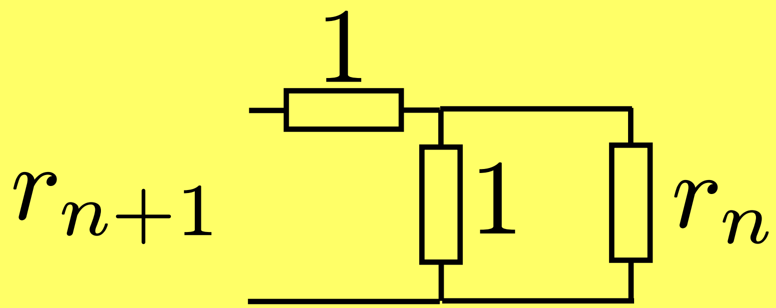


Total = $a+b$



Total = $\frac{ab}{a+b}$

Infinite Resistor Network (continued)



- $$r_{n+1} = 1 + \frac{1}{1 + \frac{1}{r_n}}$$

$$r = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} = \tau$$

Exercise on Continued Fractions (*Young*, Problem 9, page 156)

- Find $p = 2a + \cfrac{b}{2a + \frac{b}{2a + \cdots}}$

  with a, b positive integers
- $p = \lim_{n \to \infty} p_n$ with $p_1 = 2a$ and

$$p_{n+1} = 2a + \frac{b}{p_n}$$

- Define

  $p_n = u_{n+1}/u_n$, $u_1 = 1$, $u_2 = 2a$

Exercise on Continued Fractions (continued)

$$u_{n+2} - 2au_{n+1} - bu_n = 0$$

- Basis of solutions: $u_n = \lambda^n$

$$\lambda^2 - 2a\lambda - b = 0$$

- $\alpha = a + \sqrt{a^2 + b}$, $\beta = a - \sqrt{a^2 + b}$
- Note $|\beta| < a + \sqrt{a^2 + b} = \alpha$
- General solution $u_n = c\alpha^n + d\beta^n$
- $u_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ (matching u_1 and u_2)

Young, Problem 20, page 136

- For any four consecutive Fibonacci numbers $f_{n-1},\ f_n,\ f_{n+1},\ f_{n+2}$ show that $f_{n-1}f_{n+2}$ and $2f_nf_{n+1}$ form two shortest sides of a Pythagorean triangle.
- Write $f_n = b$ and $f_{n+1} = a,\ a > b$

$$a - b,\ b,\ a,\ a + b$$

- $x = a^2 - b^2,\ y = 2ab$
- $x^2 + y^2 = z^2,\ z = a^2 + b^2$

Young, Problem 20, page 136 (continued)

- Hypotenuse $z = f_n^2 + f_{n+1}^2$
- From previous class,

$$f_n^2 + f_{n+1}^2 = f_{2n+1}$$

- How is the area related to the original four numbers?

$$A = xy/2 = f_{n-1} f_n f_{n+1} f_{n+2}$$

Product of four consecutive Fibonacci numbers is the area of a Pythagorean triangle