MEASURE AND INTEGRATION: LECTURE 8

More properties of $\mathcal{L}$.

(1) All open sets and closed sets are in $\mathcal{L}$. (In particular, $\mathcal{L}$ contains the Borel σ-algebra $\mathcal{B}$.)
(2) If $\lambda^*(A) = 0$, then A is measurable and $\lambda(A) = 0$. (All sets of measure zero are measurable.)
(3) Approximation property: $A \subset \mathbb{R}^n$ is measurable $\iff$ for all $\epsilon > 0$ there exists $F \subset A \subset G$ with F closed, G open, and $\lambda(G \setminus F) < \epsilon$.

Proof. (1) G open $\Rightarrow$ $G \cap B(0,k)$ is measurable and open with $\lambda^* < \infty$. But $G = \cup_{k=1}^{\infty} G \cap B(0,k) \in \mathcal{L}$ since $\mathcal{L}$ is a σ-algebra. Moreover, again using that $\mathcal{L}$ is a σ-algebra, all closed sets are in $\mathcal{L}$.

(2) We have $0 \leq \lambda_*(A) \leq \lambda^*(A) = 0 \iff \lambda_*(A) = \lambda^*(A)$, so $A \in \mathcal{L}_0$ and $\lambda(A) = 0$.

(3) First, assume the approximation property. Thus, for each $k = 1, 2, \ldots$, there exists $F_k \subset A \subset G_k$, F_k closed, G_k open, such that $\lambda(G_k \setminus F_k) < 1/k$. Let $B = \cup_{k=1}^{\infty} F_k$. By (1) and the fact that $\mathcal{L}$ is a σ-algebra, $B \in \mathcal{L}$. Also, $B \subset A$ and $A \setminus B \subset G_k \setminus B \subset G_k \setminus F_k$. Thus,

$$\lambda^*(A \setminus B) \leq \lambda(G_k \setminus F_k) < 1/k.$$

Since this holds for any k, $\lambda^*(A \setminus B) = 0$. Thus, $A \setminus B \in \mathcal{L}$ and $\lambda(A \setminus B) = 0$. But $A = (A \setminus B) \cup B$, so $A \in \mathcal{L}$.

For the converse, assume $A \in \mathcal{L}$ and let $\epsilon > 0$ be given. Let $E_k = B(0,k) \setminus B(0,k-1) = \{x \in \mathbb{R}^n \mid k-1 \leq \|x\| < k\}$. Then $E_k \in \mathcal{L}_0$, so $A \cap E_k \in \mathcal{L}_0$. By the approximation property for $\mathcal{L}_0$, there exist $K_k \subset A \cap E_k \subset G_k$ such that $\lambda(G_k \setminus K_k) < \epsilon/2^k$. Let $F = \cup_{k=1}^{\infty} K_k$ and $G = \cup_{k=1}^{\infty} G_k$. Since arbitrary unions of open sets are open, G is open. Though this is not true for arbitrary unions of closed sets, F is nevertheless closed. (Proof: Let x be a limit point of F. Then x has to be a limit point of some K_k,

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and since each K_k is closed, $x \in K_k$.) Now $F \subset A \subset G$ and

$$G \setminus F = \left(\bigcup_{k=1}^{\infty} G_k \right) \setminus F = \bigcup_{k=1}^{\infty} (G_k \setminus F) \subset \bigcup_{k=1}^{\infty} (G_k \setminus F_k).$$

Hence,

$$\lambda(G \setminus F) \leq \sum_{k=1}^{\infty} \lambda(G_k \setminus K_k) < \epsilon \sum_{k=1}^{\infty} 2^{-k} = \epsilon.$$

$\square$

(4) If $A \in \mathcal{L}$ and $\lambda^*(A) < \infty$, we have that $\lambda_*(A) = \lambda^*(A) = \lambda(A)$. We claim this is true even if $\lambda^*(A) = \infty$. If $\lambda(A) < \infty$, then by the approximation property there exist $F \subset A \subset G$ such that $\lambda(G \setminus F) < 1$. Then

$$\lambda(G) = \lambda(G \setminus A) + \lambda(A) \leq \lambda(G \setminus F) + \lambda(A) < 1 + \lambda(A) < \infty.$$

This is a contradiction; it must be that $\lambda(A) = \infty$.

Now consider

$$A \cap B(0, 1) \subset A \cap B(0, 2) \subset A \cap B(0, 3) \subset \cdots .$$

Then

$$\lambda(A) = \lim_{k \to \infty} \lambda(A \cap B(0, k)) = \infty.$$

Since $A \cap B(0, k) \in \mathcal{L}_0$ for each k,

$$\lambda(A \cap B(0, k)) = \lambda_*(A \cap B(0, k)) \leq \lambda_*(A),$$

and so $\lambda_*(A) = \infty$.

(6) If $A \subset B$ and B is measurable, then $\lambda^*(A) + \lambda_*(B \setminus A) = \lambda(B)$.

Let G be an open set such that $G \supset A$. Then

$$\begin{aligned} \lambda(G) + \lambda_*(B \setminus A) &\geq \lambda(B \cap G) + \lambda_*(B \setminus A) \\ &\geq \lambda(B \cap G) + \lambda_*(B \setminus G) \\ &= \lambda(B \cap G) + \lambda(B \setminus G) = \lambda(B). \end{aligned}$$

Since G is arbitrary, $\lambda^*(A) + \lambda_*(B \setminus A) \geq \lambda(B)$. Next take $K \subset B \setminus A$ compact. Then

$$\begin{aligned} \lambda^*(A) + \lambda(K) &\leq \lambda^*(B \setminus K) + \lambda(K) \\ &= \lambda(B \setminus K) + \lambda(K) = \lambda(B). \end{aligned}$$

Since K is arbitrary, $\lambda^*(A) + \lambda_*(B \setminus A) \leq \lambda(B)$.

(7) (Carathéodory condition) A set A is measurable if and only if for every set $E \subset \mathbb{R}^n$,

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \cap A^c).$$

If $A \in \mathcal{L}$, then let $G \supset E$ be open. Then

$$\lambda(G) = \lambda(G \cap A) + \lambda(G \cap A^c) \geq \lambda^*(E \cap A) + \lambda^*(E \cap A^c).$$

Since G is arbitrary,

$$\lambda^*(E) \geq \lambda^*(E \cap A) + \lambda^*(E \cap A^c),$$

but

$$\lambda^*(E) \leq \lambda^*(E \cap A) + \lambda^*(E \cap A^c)$$

by subadditivity.

Conversely, let $E, M \in \mathcal{L}_0$. We have assumed that $\lambda^*(M) = \lambda^*(M \cap A)\lambda^*(M \cap A^c)$. From (6), since $M \cap A^c \subset M$,

$$\begin{aligned} \lambda(M) &= \lambda^*(M \cap A^c) + \lambda_*(M \setminus (M \cap A^c)) \\ &= \lambda^*(M \cap A^c) + \lambda_*(M \cap A). \end{aligned}$$

Thus, $\lambda^*(M \cap A) = \lambda_*(M \cap A)$, and so $A \cap M \in \mathcal{L}_0$. Since M is arbitrary, $A \in \mathcal{L}$.

Discussion. The Carathéodory condition is quite significant. It shows that the knowledge of the properties of outer measure alone is sufficient to decide which sets are measurable. Although this means that the Lebesgue measure could have been developed entirely in terms of $\lambda(I)$ for special rectangles, our method of development is preferable for the beginner.