MATH 18.152 - PROBLEM SET 10

18.152 Introduction to PDEs, Fall 2011   Professor: Jared Speck

Problem Set 10, **Due: at the start of class on 12-1-11**

I. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth, compactly supported function. Let $u : \mathbb{R}^{1+n} \to \mathbb{R}$ be the corresponding smooth solution to the global Cauchy problem for the heat equation on $\mathbb{R}^{1+n}$:

$$(0.0.1) \qquad -\partial_t u(t,x) + \Delta u(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^n,$$

$$(0.0.2) \qquad u(0,x) = f(x), \qquad x \in \mathbb{R}^n.$$

a) Let

$$(0.0.3) \qquad \hat{u}(t,\xi) \overset{\text{def}}{=} \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} u(t,x) \, d^n x$$

be the spatial Fourier transform of u. Show that for each fixed ξ, $\hat{u}(t,\xi)$ is a solution to the ODE

$$(0.0.4) \qquad \partial_t \hat{u}(t,\xi) = -4\pi^2 |\xi|^2 \hat{u}(t,\xi)$$

with initial conditions

$$(0.0.5) \qquad \hat{u}(0,\xi) = \hat{f}(\xi).$$

b) Integrate the ODE (0.0.4) and use (0.0.5) to conclude that for $t \geq 0$, we have

$$(0.0.6) \qquad \hat{u}(t,\xi) = \hat{f}(\xi) e^{-4t\pi^2 |\xi|^2}.$$

c) Let

$$(0.0.7) \qquad g(\xi) \overset{\text{def}}{=} e^{-4t\pi^2 |\xi|^2}$$

denote the function appearing on the right-hand side of (0.0.6). Show that

$$(0.0.8) \qquad g^{\vee}(x) = \frac{1}{(4\pi t)^{n/2}} e^{-x^2/(4t)} \overset{\text{def}}{=} \Gamma(t,x).$$

d) Finally, using the basic properties of the Fourier transform, show that for any $t > 0$, we have

$$(0.0.9) \qquad u(t,x) = (\Gamma(t,\cdot) * f(\cdot))(x),$$

where $*$ denotes convolution.

Remark 0.0.1. $\Gamma(t, x)$ is exactly the fundamental solution to the heat equation when the diffusion constant D is equal to 1. Equation (0.0.9) is the same convolution formula for the solution to the heat equation that we derived earlier in the course. Thus, the the above argument, which heavily depends on the properties of the Fourier transform, offers us an alternative derivation of the fundamental solution for the heat equation.

II. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth, compactly supported function.

a) Show that

$$(0.0.10) \qquad\qquad \int_{\mathbb{R}} |f(x)|^2 \, dx = - \int_{\mathbb{R}} 2x f(x) f'(x) \, dx.$$

Hint: Consider the identity $|f(x)|^2 = |f(x)|^2 \frac{d}{dx} x$ together with integration by parts.

b) Use part **a)** to show that

$$(0.0.11) \qquad\qquad \|f\|_{L^2}^2 \leq 2\|xf\|_{L^2} \|f'\|_{L^2}.$$

c) Use part **b)**, basic properties properties of the Fourier transform, and Plancherel's theorem to conclude that

$$(0.0.12) \qquad\qquad \|f\|_{L^2}^2 \leq 4\pi \|xf\|_{L^2} \|\xi \hat{f}\|_{L^2}.$$

Remark 0.0.2. Inequality (0.0.12) is one version of *Heisenberg's uncertainty principle*. Roughly speaking, it shows that f and $\hat{f}$ cannot simultaneously be localized around the origin (because a simultaneous localization would force the right-hand side of (0.0.12) to be smaller than it is allowed to be). With a little more work, it can be shown that f and $\hat{f}$ cannot simultaneously be localized around any point.

III. Let $f : \mathbb{R} \to \mathbb{C}$ be a smooth, compactly supported function.
a) Let $R > 0$ be a real number. Show that the series

$$(0.0.13) \qquad\qquad \sum_{m=0}^{\infty} f(x) \frac{(-2\pi i x)^m}{m!} \xi^m$$

converges absolutely and uniformly for $x \in (-\infty, \infty)$, $\xi \in (-R, R)$ to

$$(0.0.14) \qquad\qquad f(x) e^{-2\pi i \xi x}.$$

b) By part **a)**, we have that

$$(0.0.15) \qquad\qquad \hat{f}(\xi) = \int_{\mathbb{R}} \left\{ \sum_{m=0}^{\infty} f(x) \frac{(-2\pi i x)^m}{m!} \xi^m \right\} dx.$$

Show that we can integrate (0.0.15) term by term to conclude that

$$(0.0.16) \qquad \hat{f}(\xi) = \sum_{m=0}^{\infty} \xi^m \int_{\mathbb{R}} f(x) \frac{(-2\pi i x)^m}{m!} \, dx.$$

Then explain why the series (0.0.16) converges absolutely and uniformly for $\xi \in (-R, R)$.

Remark 0.0.3. In particular, this shows that $\hat{f}(\xi)$ is analytic (i.e., it has a convergent power series representation) for $\xi \in (-R, R)$. Since $R > 0$ was arbitrary, this shows that $\hat{f}(\xi)$ is analytic on all of $\mathbb{R}$.

c) Briefly explain why $\hat{f}(\xi)$ cannot be compactly supported unless $\hat{f}(\xi) \equiv 0$.

Remark 0.0.4. In fact, more is true: $\hat{f}(\xi)$ cannot vanish on any open set.

Hint: How many analytic functions are there that vanish on an open set?

IV. Let $\phi : \mathbb{R}^n \to \mathbb{C}$ be a smooth, compactly supported function. Let $f : \mathbb{R}^{1+n} \to \mathbb{C}$ be a smooth function such that for each fixed t, $f(t,x)$ is a compactly supported function of x. Consider the following global Cauchy problem for an inhomogeneous Schrödinger equation:

$$(0.0.17a) \qquad i\partial_t \psi(t,x) + \frac{1}{2}\Delta\psi(t,x) = f(t,x), \qquad (t,x) \in (0,\infty) \times \mathbb{R}^n,$$

$$(0.0.17b) \qquad \psi(0,x) = \phi(x), \qquad x \in \mathbb{R}^n.$$

a) Show that for each fixed ξ, $\hat{\psi}(t,\xi)$ is a solution to the ODE

$$(0.0.18) \qquad \partial_t \left(e^{i2\pi^2 t |\xi|^2} \hat{\psi}(t,\xi) \right) = -i e^{i2\pi^2 t |\xi|^2} \hat{f}(t,\xi)$$

with the initial condition

$$(0.0.19) \qquad \hat{\psi}(0,\xi) = \hat{\phi}(\xi).$$

Above, $\wedge$ denotes the Fourier transform with respect to the spatial variables.

b) Integrate (0.0.18) and use the initial condition to conclude that

$$(0.0.20) \qquad \hat{\psi}(t,\xi) = e^{-i2\pi^2 t |\xi|^2} \hat{\phi}(\xi) + \int_{s=0}^{t} -i e^{i2\pi^2 (t-s)|\xi|^2} \hat{f}(s,\xi) \, ds.$$

c) Use (0.0.20), the basic properties of the Fourier transform, and some results we discussed in class to formally show that

$$(0.0.21) \qquad \psi(t,x) = (K(t,\cdot) * \phi)(x) - i \int_{s=0}^{t} \left(K(t-s,\cdot) * f(s,\cdot) \right)(x) \, ds,$$

where

$$(0.0.22) \qquad K(t,x) \overset{\text{def}}{=} \frac{1}{(2\pi i t)^{n/2}} e^{i\frac{|x|^2}{2t}}$$

is the fundamental solution to the free Schrödinger equation.

Remark 0.0.5. Equation (0.0.21) is a version of the Duhamel principle that we introduced earlier in the course.

18.152 Introduction to Partial Differential Equations.
Fall 2011