MATH 18.152 COURSE NOTES - CLASS MEETING # 5

18.152 Introduction to PDEs, Fall 2011 Professor: Jared Speck

Class Meeting # 5: The Fundamental Solution for the Heat Equation

1. The Fundamental solution

As we will see, in the case $\Omega = \mathbb{R}^n$, we will be able to represent general solutions the inhomogeneous heat equation

$$u_t - D\Delta u = f, \qquad \Delta \overset{\text{def}}{=} \sum_{i=1}^n \partial_i^2 \tag{1.0.1}$$

in terms of f, the initial data, and a single solution that has very special properties. This special solution is called the *fundamental solution*.

Remark 1.0.1. Note that when $\Omega = \mathbb{R}^n$, there are no finite boundary conditions to worry about. However, we do have to worry about "boundary conditions at ∞." Roughly speaking, this means that we have to assume something about the growth rate of the solution as $|x| \to \infty$.

Definition 1.0.1. The fundamental solution $\Gamma_D(t,x)$ to (1.0.1) is defined to be

$$\Gamma_D(t,x) \overset{\text{def}}{=} \frac{1}{(4\pi Dt)^{n/2}} e^{-\frac{|x|^2}{4Dt}}, \qquad t > 0, x \in \mathbb{R}^n, \tag{1.0.2}$$

where $x \overset{\text{def}}{=} (x^1, \cdots, x^n)$, $|x|^2 \overset{\text{def}}{=} \sum_{i=1}^n (x^i)^2$.

Let's check that $\Gamma_D(t,x)$ solves (1.0.1) when $f = 0$ in the next lemma.

Lemma 1.0.1. *$\Gamma_D(t,x)$ is a solution to the heat equation (1.0.1) when $f = 0$ for $x \in \mathbb{R}^n, t > 0$.*

Proof. We compute that $\partial_t \Gamma_D(t,x) = \big(-\frac{2\pi D n}{(4\pi Dt)^{n/2+1}} + \frac{1}{(4\pi Dt)^{n/2}} \frac{|x|^2}{4Dt^2} \big) e^{-\frac{|x|^2}{4Dt}}$. Also, we compute $\partial_i \Gamma_D(t,x) = -\frac{2\pi x^i}{(4\pi Dt)^{n/2+1}} e^{-\frac{|x|^2}{4Dt}}$ and $\partial_i^2 \Gamma_D(t,x) = \big(-\frac{2\pi}{(4\pi Dt)^{n/2+1}} + \frac{1}{4Dt} \frac{2\pi (x^i)^2}{(4\pi Dt)^{n/2+1}} \big) e^{-\frac{|x|^2}{4Dt}}$,
$D\Delta \Gamma_D(t,x) = \big(-\frac{2\pi D n}{(4\pi Dt)^{n/2+1}} + \frac{1}{4Dt} \frac{2\pi D |x|^2}{(4\pi Dt)^{n/2+1}} \big) e^{-\frac{|x|^2}{4Dt}}$. Lemma 1.0.1 now easily follows. □

Here are a few very important properties of $\Gamma_D(t,x)$.

Lemma 1.0.2. *$\Gamma_D(t,x)$ has the following properties:*

(1) *If $x \neq 0$, then* $\lim_{t \to 0^+} \Gamma_D(t,x) = 0$
(2) $\lim_{t \to 0^+} \Gamma_D(t,0) = \infty$
(3) $\int_{\mathbb{R}^n} \Gamma_D(t,x) \, d^n x = 1$ *for all $t > 0$*

Proof. This is a good exercise for you to do on your own. □

As we will see, (1) - (3) suggest that at $t = 0$, $\Gamma_D(0,x)$ behaves like the "delta distribution centered at 0." We'll make sense of this in the next lemma.

Remark 1.0.2. The delta distribution is sometimes called the "delta function," but it is *not a function in the usual sense*!

So what is the delta distribution?

Definition 1.0.2. The delta distribution δ is an example of a mathematical object called a *distribution.* It acts on suitable functions $\phi(x)$ as follows:

$$\langle \delta, \phi \rangle \stackrel{\text{def}}{=} \phi(0). \tag{1.0.3}$$

Remark 1.0.3. The notation $\langle \cdot, \cdot \rangle$ is meant to remind you of the L^2 inner product

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x) g(x)\, d^n x. \tag{1.0.4}$$

The next lemma shows that $\Gamma_D(t,x)$ behaves like the delta distribution as $t \to 0^+$.

Lemma 1.0.3. *Suppose that $\phi(x)$ is a continuous function on $\mathbb{R}^n$ and that there exist constants $a, b \geq 0$ such that*

$$|\phi(x)| \leq a e^{b|x|^2}. \tag{1.0.5}$$

Then

$$\lim_{t \to 0^+} \int_{\mathbb{R}^n} \Gamma_D(t,x) \phi(x)\, d^n x = \phi(0). \tag{1.0.6}$$

Proof. Using Property (3) of Lemma 1.0.2, we start with the simple inequality

$$\phi(0) = \int_{\mathbb{R}^n} \Gamma_D(t,x) \phi(0)\, d^n x = \int_{\mathbb{R}^n} \Gamma_D(t,x)\phi(x)\, d^n x + \int_{\mathbb{R}^n} \Gamma_D(t,x)(\phi(0) - \phi(x))\, d^n x. \tag{1.0.7}$$

Let $\epsilon > 0$ be any small positive number, and choose a ball B of radius R centered at 0 such that $|\phi(0) - \phi(x)| \leq \epsilon$ for $x \in B$ (this is possible since ϕ is continuous). Then the last term from above can be estimated as follows, where B^c denotes the complement of B in $\mathbb{R}^n$:

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \Gamma_D(t,x)(\phi(0) - \phi(x))\, d^n x \right| &\leq \int_B \Gamma_D(t,x) |\phi(0) - \phi(x)|\, d^n x + \int_{B^c} \Gamma_D(t,x) |\phi(0) - \phi(x)|\, d^n x \\ &\leq \int_B \Gamma_D(t,x) \epsilon\, d^n x + |\phi(0)| \int_{B^c} \Gamma_D(t,x)\, d^n x + \int_{B^c} \Gamma_D(t,x) |\phi(x)|\, d^n x \\ &\leq \epsilon + |\phi(0)| \int_{B^c} \Gamma_D(t,x)\, d^n x + \int_{B^c} \Gamma_D(t,x)|\phi(x)|\, d^n x. \end{aligned} \tag{1.0.8}$$

We have thus shown that

$$\left| \phi(0) - \int_{\mathbb{R}^n} \Gamma_D(t,x)\phi(x)\, d^n x \right| \leq \epsilon + |\phi(0)| \int_{B^c} \Gamma_D(t,x)\, d^n x + \int_{B^c} \Gamma_D(t,x) |\phi(x)|\, d^n x. \tag{1.0.9}$$

To estimate the final term on the right-hand side of (1.0.8), we take advantage of the spherical symmetry of $\Gamma(t,x)$ in x. More precisely, we introduce the radial variable $r = |x| \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^n (x^i)^2}$

and recall from vector calculus that for spherically symmetric functions, $d^n x = C_n r^{n-1}\, dr$ where $C_n > 0$ is a constant. Therefore, using the assumed bound $|\phi(x)| \leq a e^{br^2}$, we have that

(1.0.10)
$$\int_{B^c} \Gamma_D(t,x)|\phi(x)|\, d^n x \leq C_n' t^{-n/2} \int_{r=R}^{\infty} r^{n-1} e^{-(\frac{1}{4Dt}-b)r^2}\, dr \leq C_n'' \Big(\frac{1}{4D} - bt\Big)^{-n/2} \int_{\rho=R\sqrt{\frac{1}{4Dt}-b}}^{\infty} \rho^{n-1} e^{-\rho^2}\, d\rho,$$

where $C_n' > 0$ and $C_n'' > 0$ are constants. To deduce the second inequality in (1.0.10), we have made the change of variables $r = \rho(\frac{1}{4Dt} - b)^{-1/2} = \rho t^{1/2}(\frac{1}{4D} - bt)^{-1/2}$. Now since $R\sqrt{\frac{1}{4Dt} - b} \to \infty$ as $t \to 0^+$, it easily follows from the last expression in (1.0.10) that $\int_{B^c} \Gamma_D(t,x)\, d^n x$ goes to 0 as $t \to 0^+$.

The second term on the right-hand side of (1.0.8) can similarly be shown to go to 0 as $t \to 0^+$.

Combining the above arguments, we have thus shown that for any $\epsilon > 0$,

$$\limsup_{t \to 0^+} \Big| \phi(0) - \int_{\mathbb{R}^n} \Gamma_D(t,x)\phi(x)\, d^n x \Big| \leq \epsilon. \tag{1.0.11}$$

We therefore conclude that

$$\lim_{t \to 0^+} \Big| \phi(0) - \int_{\mathbb{R}^n} \Gamma_D(t,x)\phi(x)\, d^n x \Big| = 0 \tag{1.0.12}$$

as desired.

□

Remark 1.0.4. Lemma 1.0.3 can be restated as

$$\lim_{t \to 0^+} \langle \Gamma_D(t,\cdot), \phi(\cdot) \rangle = \langle \delta(\cdot), \phi(\cdot) \rangle = \phi(0). \tag{1.0.13}$$

On the left, $\langle , \rangle$ means the integral inner product, whereas in the middle it has the meaning of (1.0.3). We sometimes restate (1.0.13) as

$$\lim_{t \to 0^+} \Gamma_D(t,x) = \delta(x). \tag{1.0.14}$$

Let's summarize the above results.

Proposition 1.0.4 (Properties of $\Gamma_D(t,x)$). $\Gamma_D(t,x)$ *is a solution to the heat equation* (1.0.1) *(with $f = 0$) verifying the initial conditions*

$$\lim_{t \to 0^+} \Gamma_D(t,x) = \delta(x). \tag{1.0.15}$$

1.1. **Solving the global Cauchy problem when $n = 1$.** Let's see how we can use Γ_D to solve the following initial value (aka Cauchy) problem:

$$\begin{aligned} u_t - D u_{xx} &= 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) &= g(x). \end{aligned} \tag{1.1.1}$$

We will make use of an important mathematical operation called *convolution*.

Definition 1.1.1. If f and g are two functions on $\mathbb{R}^n$, then we define their convolution $f * g$ to be the following function on $\mathbb{R}^n$:

$$(f * g)(x) \overset{\text{def}}{=} \int_{\mathbb{R}^n} f(y)g(x-y)\,d^n y. \tag{1.1.2}$$

Convolution is an averaging process, in which the function $f(x)$ is replaced by the "average value" of $f(x)$ relative to the "profile" function $g(x)$.

The convolution operator plays a very important role in many areas of mathematics. Here are two key properties. First, by making the change of variables $z = x - y$, $d^n z = d^n y$ in (1.1.2), we see that

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,d^n y = \int_{\mathbb{R}^n} f(x-z)g(z)\,d^n z = (g * f)(x), \tag{1.1.3}$$

which implies that convolution is a *commutative* operation. Next, Fubini's theorem can be used to show that

$$f * (g * h) = (f * g) * h, \tag{1.1.4}$$

so that $*$ is also associative.

Remark 1.1.1. According to (1.0.3) and (1.1.3),

$$(f * \delta)(x) = \langle \delta(y), f(x-y)\rangle_y = f(x), \tag{1.1.5}$$

so that in the context of convolutions, the δ distribution plays the role of an "identity element."

The next proposition is a standard fact from analysis. It allows us to differentiate under integrals under certain assumptions. We will use it in the proof of the next theorem.

Proposition 1.1.1 (Differentiating under the integral). *Let $I(a,b)$ be a function on $\mathbb{R} \times \mathbb{R}$. Assume that*

$$\int_{\mathbb{R}} |I(a,b)|\,da < \infty \tag{1.1.6}$$

for all b belonging to a neighborhood of b_0 and define

$$J(b) \overset{\text{def}}{=} \int_{\mathbb{R}} I(a,b)\,da. \tag{1.1.7}$$

Assume that there exists a neighborhood $\mathcal{N}$ of b_0 such that for almost every[1] *a, $\partial_b I(a,b)$ exists for $b \in \mathcal{N}$. In addition, assume that there exists as function $U(a)$ (defined for almost all a) such that for $b \in \mathcal{N}$, we have that $|\partial_b I(a,b)| \leq U(a)$ and such that*

[1] In a measure theory course, you would learn a precise technical definition of "almost every." For the purposes of this course, it suffices to know the following fact: if a statement holds for all a except for those values of a belonging to a countable set, then the statement holds for almost every a. The main point is that the function $I(a,b)$ does not have to be "well-behaved" at every single value of a; it can have some "bad a spots," just not too many of them.

$$\int_{\mathbb{R}} U(a)\,da < \infty. \tag{1.1.8}$$

Then near $J(b)$ is differentiable near b_0, and

$$\partial_b J(b) = \int_{\mathbb{R}} \partial_b I(a,b)\,da. \tag{1.1.9}$$

Remark 1.1.2. An analogous proposition is true for functions $I(a,b)$ defined on $\mathbb{R}^m \times \mathbb{R}^n$.

Theorem 1.1 (Solving the global Cauchy problem via the fundamental solution). *Assume that $g(x)$ is a continuous function on $\mathbb{R}^n$ that verifies the bounds $|g(x)| \leq ae^{b|x|^2}$, where $a,b > 0$ are constants. Then there exists a solution $u(t,x)$ to the homogeneous heat equation*

$$\begin{aligned} u_t - D\Delta u &= 0, \qquad (t > 0, x \in \mathbb{R}^n), \\ u(0,x) &= g(x), \qquad x \in \mathbb{R}^n \end{aligned} \tag{1.1.10}$$

existing for $(t,x) \in [0,T) \times \mathbb{R}^n$, where

$$T \overset{def}{=} \frac{1}{4Db}. \tag{1.1.11}$$

Furthermore, $u(t,x)$ can be represented as

$$\begin{aligned} u(t,x) = [g(\cdot) * \Gamma_D(t,\cdot)](x) &= \int_{\mathbb{R}^n} g(y)\Gamma_D(t,x-y)\,d^n y \\ &= \frac{1}{(4\pi Dt)^{n/2}} \int_{\mathbb{R}^n} g(y) e^{-\frac{|x-y|^2}{4Dt}}\,d^n y. \end{aligned} \tag{1.1.12}$$

The solution $u(t,x)$ is of regularity $C^\infty\big((0,\frac{1}{4Db})\times\mathbb{R}^n\big)$ (i.e., it is infinitely differentiable). Finally, for each compact subinterval $[0,T'] \subset [0,T)$, there exist constants $A,B > 0$ (depending on the compact subinterval) such that

$$|u(t,x)| \leq Ae^{B|x|^2} \tag{1.1.13}$$

for all $(t,x) \in [0,T'] \times \mathbb{R}^n$. The solution $u(t,x)$ is the unique solution in the class of functions verifying a bound of the form (1.1.13).

Remark 1.1.3. Note the very important **smoothing property** of diffusion: the solution to the heat equation on all of $\mathbb{R}^n$ is *smooth* even if the data are merely *continuous*.

Remark 1.1.4. The formula (1.1.12) shows that solutions to (1.1.10) propagate with **infinite speed**: even if the initial data $g(x)$ have support that is contained within some compact region, (1.1.12) shows that at any time $t > 0$, the solution $u(t,x)$ has "spread out over the entire space $\mathbb{R}^n$." In contrast, as we will see later in the course, some important PDEs have finite speeds of propagation (for example, the wave equation).

Proof. For simplicity, we only give the proof in the case $n = 1$. The basic strategy of the proof is to analyze the behavior of $\Gamma_D(t, y)$ in detail.

Let $u(t, x)$ be the function defined by (1.1.12). The argument that follows will show that the right-hand side of (1.1.12) is finite (and more). In fact, let us first demonstrate the bound (1.1.13). To this end, let $\epsilon > 0$ be any positive number. Then using the simple algebraic estimate $|2xy| \leq \epsilon^{-1}x^2 + \epsilon y^2$, we deduce the inequality

$$|x - y|^2 = x^2 - 2xy + y^2 \leq (1 + \epsilon^{-1})x^2 + (1 + \epsilon)y^2. \tag{1.1.14}$$

Using (1.1.14) and the assumed bound on $|g(\cdot)|$, we deduce that

$$|g(x - y)| \leq ae^{b|x-y|^2} \leq ae^{(1+\epsilon^{-1})b|x|^2}e^{(1+\epsilon)b|y|^2} \tag{1.1.15}$$

Using (1.1.15) and the fact that $\int_{\mathbb{R}^n} g(y)\Gamma_D(t, x - y)\,dy = \int_{\mathbb{R}^n} g(x - y)\Gamma_D(t, y)\,dy$ (i.e., that convolution is commutative), we have the following estimates:

$$\begin{aligned} |u(t, x)| &\leq \int_{\mathbb{R}} |g(x - y)|\Gamma_D(t, y)\,dy \leq ae^{(1+\epsilon^{-1})b|x|^2} \int_{\mathbb{R}} e^{(1+\epsilon)b|y|^2}\Gamma_D(t, y)\,dy \\ &\leq ae^{(1+\epsilon^{-1})b|x|^2} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi D}} t^{-1/2} e^{-\left[\frac{1}{4\pi Dt} - (1+\epsilon)b\right]y^2}\,dy \\ &= ae^{(1+\epsilon^{-1})b|x|^2} \frac{1}{\sqrt{4\pi D}} \Big[\frac{1}{4\pi D} - (1 + \epsilon)bt\Big]^{-1/2} \underbrace{\int_{\mathbb{R}} e^{-z^2}\,dz}_{<\infty} \\ &\leq Ae^{(1+\epsilon^{-1})b|x|^2}, \end{aligned} \tag{1.1.16}$$

where $A > 0$ is an $\epsilon-$dependent constant, and in the next-to-last step, we have made the change of variables $z = \big[\frac{1}{4\pi Dt} - (1 + \epsilon)b\big]^{1/2} y = t^{-1/2}\big[\frac{1}{4\pi D} - (1 + \epsilon)bt\big]^{1/2} y$. Note that this change of variables is valid as long as $0 < t < \frac{1}{4\pi D(1+\epsilon)b}$. Since ϵ is allowed to be arbitrarily small, we have thus demonstrated an estimate of the form (1.1.13).

Let's now check that the function $u(t, x)$ defined by (1.1.12) is a solution to the heat equation and also that it takes on the initial conditions $g(x)$. To this end, let $\mathcal{L} \stackrel{\text{def}}{=} \partial_t - D\partial_x^2$. We want to show that $\mathcal{L}u(t, x) = 0$ for $t > 0, x \in \mathbb{R}^n$ and that $u(t, x) \to g(x)$ as $t \downarrow 0$. Recall that by Proposition 1.0.4, $\mathcal{L}\Gamma_D(t, x) = 0$ for $t > 0, x \in \mathbb{R}$. For $t > 0, x \in \mathbb{R}$, we have that

$$\mathcal{L}u(t, x) = \int_{\mathbb{R}} g(y)\overbrace{\mathcal{L}\Gamma_D(t, x - y)}^{0}\,dy = 0. \tag{1.1.17}$$

To derive (1.1.17), we have used Proposition 1.1.1 to differentiate under the integral; because of rapid exponential decay of $\Gamma_D(\cdot, \cdot)$ in its second argument as the argument goes to ∞, one can use arguments similar to those given in the beginning of this proof to check that the hypotheses of the proposition are verified.

Similarly, the fact that $u \in C^\infty\big((0, \frac{1}{4Db}) \times \mathbb{R}\big)$ can be derived by repeatedly differentiating with respect to t and x under the integral in (1.1.12).

Furthermore by (1.0.15) and (1.1.5), we have that

$$\lim_{t \to 0^+} u(t,x) = \lim_{t \to 0^+} (g(\cdot) * \Gamma_D(t,\cdot))(x) = (g * \delta)(x) = g(x). \tag{1.1.18}$$

The question of uniqueness in the class of solutions verifying a bound of the form (1.1.13) is challenging and will not be addressed here. Instead, with the help of the weak maximum principle, you will prove a weakened version of the uniqueness result in your homework.

□

In the next theorem, we extend the results of Theorem 1.1 to allow for an inhomogeneous term $f(t,x)$.

Theorem 1.2 (**Duhamel's principle**). *Let $g(x)$ and $T \overset{def}{=} \frac{1}{4Db}$ be as in Theorem 1.1. Also assume that $f(t,x)$, $\partial_i f(t,x)$, and $\partial_i\partial_j f(t,x)$ are continuous, bounded functions on $[0,T) \times \mathbb{R}^n$ for $1 \leq i,j \leq n$. Then there exists a unique solution $u(t,x)$ to the inhomogeneous heat equation*

$$\begin{aligned} u_t - D\Delta u &= f(t,x), \qquad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) &= g(x), \qquad x \in \mathbb{R} \end{aligned} \tag{1.1.19}$$

existing for $(t,x) \in [0,T) \times \mathbb{R}$. Furthermore, $u(t,x)$ can be represented as

$$u(t,x) = (\Gamma_D(t,\cdot) * g)(x) + \int_0^t (\Gamma_D(t-s,\cdot) * f(s,\cdot))(x)\, ds. \tag{1.1.20}$$

The solution has the following regularity properties: $u \in C^0([0,T) \times \mathbb{R}) \cap C^{1,2}((0,T) \times \mathbb{R})$.

Proof. A slightly less technical version of this theorem is one of your homework exercises. □

2. Deriving $\Gamma_D(t,x)$

Let's backtrack a bit and discuss how one could derive the fundamental solution to the heat equation

$$\partial_t u(t,x) - D\Delta_x u(t,x) = 0, \qquad (t,x) \in [0,\infty) \times \mathbb{R}^n. \tag{2.0.21}$$

As we will see, the fundamental solution is connected to some important invariance properties associated to solutions of (2.0.21). These properties are addressed in the next lemma.

Lemma 2.0.2 (**Invariance of solutions to the heat equation under translations and parabolic dilations**). *Suppose that $u(t,x)$ is a solution to the heat equation* (2.0.21). *Let $A, t_0 \in \mathbb{R}$ be constants, and $x_0 \in \mathbb{R}^n$. Then the amplified and translated function*

$$u^*(t,x) \overset{def}{=} Au(t-t_0, x-x_0) \tag{2.0.22}$$

is also a solution to (2.0.21).

Similarly, if $\lambda > 0$ is a constant, then the amplified, parabolically scaled function

$$u^*(t,x) \overset{def}{=} Au(\lambda^2 t, \lambda x) \tag{2.0.23}$$

is also a solution.

Proof. We address only the case (2.0.23), and leave (2.0.22) as a simple exercise. Using the chain rule, we calculate that if u is a solution to (2.0.21), then

$$\partial_t u^*(t,x) - \Delta u^*(t,x) = \lambda^2 A\Big\{(\partial_t u)(\lambda^2 t, \lambda x) - (D\Delta u)(\lambda^2 t, \lambda x)\Big\} = 0. \tag{2.0.24}$$

Thus, u^* is also a solution. □

We would now like to choose the constant A in (2.0.23) so that the "total thermal energy" of u^* is equal to the "total thermal energy of" of u.

Definition 2.0.2. We define the *total thermal energy* $\mathcal{T}(t)$ at time t associated to $u(t,x)$ by

$$\mathcal{T}(t) \overset{\text{def}}{=} \int_{\mathbb{R}^n} u(t,x)\, d^n x. \tag{2.0.25}$$

It is important to note that for rapidly-spatially decaying solutions to the heat equation, $\mathcal{T}(t)$ is constant.

Lemma 2.0.3. *Let $u(t,x) \in C^{1,2}([0,\infty)\times\mathbb{R}^n)$ be a solution to the heat equation $-\partial_t u(t,x) + \Delta u(t,x) = 0$. Assume that at each fixed t, $\lim_{|x|\to\infty} |x|^{n-1}|\nabla_x u(t,x)| = 0$, uniformly in x. Furthermore, assume that there exists a function $f(x) \geq 0$, not depending on t, such that $|\partial_t u| \leq f(x)$ and such that $\int_{\mathbb{R}^n} f(x)\, d^n x < \infty$. Then the total thermal energy of $u(t,x)$ is constant in time:*

$$\mathcal{T}(t) = \mathcal{T}(0). \tag{2.0.26}$$

Proof. Let $\mathcal{T}(t) \overset{\text{def}}{=} \int_{\mathbb{R}^n} u(t,x)\, d^n x$ denote the total thermal energy at time t. The hypotheses on ensure that we can differentiate under the integral and use the heat equation:

$$\frac{d}{dt}\mathcal{T}(t) = \int_{\mathbb{R}^n} \partial_t u(t,x)\, d^n x = \int_{\mathbb{R}^n} \Delta u(t,x)\, d^n x = \lim_{R\to\infty}\int_{B_R(0)} \Delta u(t,x)\, d^n x, \tag{2.0.27}$$

where $B_R(0) \subset \mathbb{R}^n$ denotes the ball of radius R centered at the origin. Then with the help of the divergence theorem, and recalling that $d\sigma = R^{n-1} d\omega$ along $\partial B_R(0)$, where ω denotes angular coordinates along the unit sphere $\partial B_1(0)$, we conclude that

$$\begin{aligned}
\lim_{R\to\infty}\int_{B_R(0)} \Delta u(t,x)\, d^n x &= \lim_{R\to\infty}\int_{\partial B_R(0)} \nabla_{\hat{N}} u(t,\sigma)\, d\sigma \\
&= \lim_{R\to\infty}\int_{\partial B_1(0)} R^{n-1}\nabla_{\hat{N}} u(t,R\omega)\, d\omega \\
&= \int_{\partial B_1(0)} \lim_{R\to\infty} R^{n-1}\nabla_{\hat{N}} u(t,R\omega)\, d\omega = \int_{\partial B_1(0)} 0\, d\omega = 0.
\end{aligned} \tag{2.0.28}$$

In the last steps, we have used the following basic fact from analysis: the condition $\lim_{|x|\to\infty}|x|^{n-1}|\nabla_x u(t,x)| = 0$ uniformly in ω allows us to interchange the order of the limit and the integral. □

We now return to the issue of choosing constant A in (2.0.23) so that the total thermal energy of u^* is equal to the total thermal energy of of u. Using the change of variables $z = \lambda x$, and recalling from multi-variable calculus that $d^n z = \lambda^n d^n x$, we compute that

(2.0.29)
$$\int_{\mathbb{R}^n} u^*(t,x)\,d^n x = A\int_{\mathbb{R}^n} u(D^2\lambda^2 t, \lambda x)\,d^n x = A\lambda^{-n}\int_{\mathbb{R}^n} u(\lambda^2 t, z)\,d^n z.$$

Observe that that $\int_{\mathbb{R}^n} u(\lambda^2 t, z)\,d^n z$ is in fact the mass of u. Thus, we choose $A = \lambda^n$, which results in

(2.0.30)
$$u^*(t,x) = \lambda^n u(D^2\lambda^2 t, \lambda x).$$

Motivated by the parabolic scaling result (2.0.23), we now introduce the dimensionless variable

(2.0.31)
$$\zeta \overset{\text{def}}{=} \frac{x}{\sqrt{Dt}},$$

where we have used the fact that the constant D has the dimensions of [length^2]/[time]. Note that ζ is *invariant* under the parabolic scaling $t \to \lambda^2 t$, $x \to \lambda x$.

We now proceed to derive the fundamental solution. For simplicity, we only consider the case of $1+1$ spacetime dimensions. We will look for a fundamental solution of the form

(2.0.32)
$$\Gamma_D(t,x) = \frac{1}{\sqrt{Dt}} V(\zeta),$$

where $V(\zeta)$ is a function that we hope to determine. Admittedly, it is not easy to completely motivate the fact that $\Gamma_D(t,x)$ should look like (2.0.32). We first note that since we would like to achieve $\int_{\mathbb{R}} \Gamma_D(t,x) = 1$, the change of variables (2.0.31) leads to the following identity:

(2.0.33)
$$1 = \int_{\mathbb{R}} \Gamma_D(t,x) \int_{\mathbb{R}} \frac{1}{\sqrt{Dt}} V\Big(\frac{x}{\sqrt{Dt}}\Big)\,dx = \int_{\mathbb{R}} V(\zeta)\,d\zeta.$$

Next, since $\Gamma_D(t,x)$ is assumed to solve the heat equation, we calculate that

(2.0.34)
$$0 = \partial_t \Gamma - \Delta\Gamma = -\frac{1}{2} D^{-1/2} t^{-3/2} \Big\{ V''(\zeta) + \frac{1}{2}\zeta V'(\zeta) + \frac{1}{2} V(\zeta) \Big\}.$$

Therefore, V must be a solution to the following ODE:

(2.0.35)
$$V''(\zeta) + \frac{1}{2}\zeta V'(\zeta) + \frac{1}{2} V(\zeta) = 0.$$

Since we want $\Gamma_D(t,x)$ to behave like the δ distribution (at least for small $t > 0$), we demand that

(2.0.36)
$$V(\zeta) \geq 0.$$

Furthermore, since we want $\Gamma_D(t,x)$ to rapidly decay as $|x| \to \infty$, we demand that

(2.0.37)
$$V(\pm\infty) = 0.$$

We also expect that ideally, $V(\zeta)$ should be an even function. Furthermore, it is easy to see that if $V(\zeta)$ is a solution to (2.0.35), then so is $W(\zeta) \overset{\text{def}}{=} V(-\zeta)$. Thus, it is reasonable to look for an even solution. Now for any differentiable even function $V(\zeta)$, it necessarily follows that $V'(0) = 0$. Thus, we demand that

$$V'(0) = 0. \tag{2.0.38}$$

We now note that (2.0.35) can be written in the form

$$\frac{d}{d\zeta}(V'(\zeta) + \frac{1}{2}\zeta V(\zeta)) = 0, \tag{2.0.39}$$

which implies that $V'(\zeta) + \frac{1}{2}\zeta V(\zeta)$ is constant. By setting $\zeta = 0$ in and using (2.0.38), we see that this constant is 0 :

$$V'(\zeta) + \frac{1}{2}\zeta V(\zeta) = 0. \tag{2.0.40}$$

Now the first-order ODE (2.0.40) can be written in the form

$$\frac{d}{d\zeta}\ln V(\zeta) = -\frac{1}{2}\zeta, \tag{2.0.41}$$

which can be easily integrated as follows:

$$\ln\left(\frac{V(\zeta)}{V(0)}\right) = -\frac{1}{4}\zeta^2, \tag{2.0.42}$$

$$\implies V(\zeta) = V(0)e^{-\frac{1}{4}\zeta^2}. \tag{2.0.43}$$

To find $V(0)$, we use the relation (2.0.33), and the integral identity[2]

$$1 = \int_{\mathbb{R}} V(0)e^{-\frac{1}{4}\zeta^2}\,d\zeta \underset{\zeta=2\alpha}{=} 2V(0)\int_{\mathbb{R}} e^{-\alpha^2}\,d\alpha = 2V(0)\sqrt{\pi}. \tag{2.0.44}$$

Therefore, $V(0) = \frac{1}{\sqrt{4\pi}}$, and

$$V(\zeta) = \frac{1}{\sqrt{4\pi}}e^{-\frac{1}{4}\zeta^2}. \tag{2.0.45}$$

Finally, from (2.0.32) and (2.0.45), we deduce that

$$\Gamma_D(t,x) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}} \tag{2.0.46}$$

as desired.

[2]Let $I \overset{\text{def}}{=} \int_{\mathbb{R}} e^{-x^2}\,dx$. Then $I^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(x^2+y^2)}\,dx\,dy$, and by switching to polar coordinates, we have that $I^2 = 2\pi\int_{r=0}^{\infty} re^{-r^2}\,dr = \pi$. Thus, $I = \sqrt{\pi}$.

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Fall 2011