MATH 18.152 COURSE NOTES - CLASS MEETING # 12

18.152 Introduction to PDEs, Fall 2011 Professor: Jared Speck

Class Meeting # 12: Kirchhoff's Formula and Minkowskian Geometry

1. Kirchhoff's Formula

We are now ready to derive Kirchhoff's famous formula.

Theorem 1.1 (**Kirchhoff's formula**). *Assume that $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$. Then the unique solution $u(t,x)$ to the global Cauchy problem*

$$-\partial_t^2 u(t,x) + \Delta u(t,x) = 0, \qquad (t,x) \in [0,\infty) \times \mathbb{R}^3, \tag{1.0.1a}$$

$$u(0,x) = f(x), \qquad x \in \mathbb{R}^3, \tag{1.0.1b}$$

$$\partial_t u(0,x) = g(x), \qquad x \in \mathbb{R}^3 \tag{1.0.1c}$$

in $1+3$ dimensions satisfies $u \in C^2([0,\infty) \times \mathbb{R}^3)$ and can be represented as follows:

$$u(t,x) = \frac{1}{4\pi t^2} \int_{\partial B_t(x)} f(\sigma)\, d\sigma + \frac{1}{4\pi t} \int_{\partial B_t(x)} \nabla_{\hat{N}(\sigma)} f(\sigma)\, d\sigma + \frac{1}{4\pi t} \int_{\partial B_t(x)} g(\sigma)\, d\sigma. \tag{1.0.2}$$

Remark 1.0.1. Equation (1.0.2) again illustrates the finite speed of propagation property associated to the linear wave equation. More precisely, the behavior of the solution at the point (t,x) is only affected by the initial data in the region $\{(0,y) \mid |x-y| = t\}$. The fact that this region is the *boundary* of a ball rather than a solid ball is known as the *sharp Huygens principle.* It can be shown that the sharp version of this principle holds $1+n$ dimensions when $n \geq 3$ is odd, but *not when $n = 1$ or when n is even.* However, even when the sharp version fails, there still is a finite speed of propagation property; the solution in these cases depends on the data in the solid ball.

Remark 1.0.2. Note that in Theorem 1.1, we can only guarantee that the solution is one degree *less differentiable* than the data. This contrasts to d'Alembert's formula, in which the $1+1$ dimensional solution was shown to have the same degree of differentiability as the data.

Proof. Using the **Representation formula for** $\widetilde{U}(t,r;x)$ corollary, the differentiability of $\widetilde{F}$, and the continuity of $\widetilde{G}$, we have that

$$\begin{aligned} u(t,x) &= \lim_{r \to 0^+} U(t,r;x) = \lim_{r \to 0^+} \frac{\widetilde{U}(t,r;x)}{r} \\ &= \lim_{r \to 0^+} \frac{\widetilde{F}(r+t;x) - \widetilde{F}(r-t;x)}{2r} + \frac{1}{2r} \int_{\rho=-r+t}^{\rho=r+t} \widetilde{G}(\rho;x)\, d\rho \\ &= \partial_t \widetilde{F}(t;x) + \widetilde{G}(t;x). \end{aligned} \tag{1.0.3}$$

The $\partial_t \widetilde{F}(t;x)$ term on the right-hand side of (1.0.3) arises from the definition of a partial derivative, while to derive the $\widetilde{G}(t;x)$ term, we applied the fundamental theorem of calculus (think about both of these claims own your own!). By the definition of $\widetilde{F}$ and $\widetilde{G}$ (see the **Spherical averages** Proposition), it therefore follows from (1.0.3) that

$$u(t,x) = \partial_t\Big(t\frac{1}{4\pi t^2}\int_{\partial B_t(x)} f(\sigma)\,d\sigma\Big) + t\frac{1}{4\pi t^2}\int_{\partial B_t(x)} g(\sigma)\,d\sigma. \tag{1.0.4}$$

Differentiating under the integral sign, using the chain rule relation $\partial_t[f(x+t\omega)] = (\nabla f)(x+t\omega)\cdot\omega = \nabla_{\hat{N}(x+t\omega)} f(x+t\omega)$ (where $\hat{N}$ is the unit outward normal to $\partial B_t(x)$), and recalling that $d\sigma = t^2 d\omega$ on $\partial B_t(x)$, we have that

$$\begin{aligned} t\partial_t\Big(\frac{1}{4\pi t^2}\int_{\partial B_t(x)} f(\sigma)\,d\sigma\Big) &= t\partial_t\Big(\frac{1}{4\pi}\int_{\partial B_1(0)} [f(x+t\omega)]\,d\omega\Big) = \frac{t}{4\pi}\int_{\partial B_1(0)} \partial_t[f(x+t\omega)]\,d\omega \\ &= \frac{t}{4\pi}\int_{\partial B_1(0)} \nabla_{\hat{N}(x+t\omega)} f(x+t\omega)\,d\omega \\ &\overset{\text{def}}{=} \frac{1}{4\pi t}\int_{\partial B_t(x)} \nabla_{\hat{N}(\sigma)} f(\sigma)\,d\sigma. \end{aligned} \tag{1.0.5}$$

Combining (1.0.4) and (1.0.5), we have that

$$u(t,x) = \frac{1}{4\pi t^2}\int_{\partial B_t(x)} f(\sigma)\,d\sigma + \frac{1}{4\pi t}\int_{\partial B_t(x)} \nabla_{\hat{N}(\sigma)} f(\sigma)\,d\sigma + \frac{1}{4\pi t}\int_{\partial B_t(x)} g(\sigma)\,d\sigma. \tag{1.0.6}$$

We have thus shown (1.0.2).

The fact that $u \in C^2([0,\infty)\times\mathbb{R}^3)$ follows from differentiating the integrals in the formula (1.0.2) and using the hypotheses on f and g.

□

Exercise 1.0.1. Show that (1.0.3) holds.

Exercise 1.0.2. Verify that $u \in C^2([0,\infty)\times\mathbb{R}^3)$, as was claimed at the end of the proof above.

The Linear Wave Equation: A Geometric Point of View

We will now derive some very important results for solutions to the linear wave equation. The results will exploit interplay between geometry and analysis. Many of the techniques that we will discuss play a central role in current PDE research.

2. Geometric background

Throughout this lecture, standard rectangular coordinates on $\mathbb{R}^{1+n}$ are denoted by $(x^0, x^1, \cdots, x^n)$, and we often use the alternate notation $x^0 = t$. The Minkowski metric on $\mathbb{R}^{1+n}$, which we denote by m, embodies the *Lorentzian geometry* at the heart of Einstein's theory of special relativity. As we will see, this geometry is intimately connected to the linear wave equation. The components of m takes the following form relative to a standard rectangular coordinate system:

$$m_{\mu\nu} = (m^{-1})^{\mu\nu} = \mathrm{diag}(-1, \underbrace{1, 1, \cdots, 1}_{n \text{ copies}}). \tag{2.0.7}$$

We can view $m_{\mu\nu}$ as an $(1+n)\times(1+n)$ matrix of real numbers. It is conventional to label the first row and column of $m_{\mu\nu}$ starting with "0" rather than "1," so that $m_{00} = -1$, $m_{22} = 1$, $m_{02} = 0$, etc. Note that m is symmetric: $m_{\mu\nu} = m_{\nu\mu}$.

If X is a vector in $\mathbb{R}^{1+n}$ with components X^μ $(0 \leq \mu \leq n)$, then we define its metric dual to be the covector with components X_μ $(0 \leq \mu \leq n)$ defined by

$$X_\mu \overset{\text{def}}{=} \sum_{\alpha=0}^{3} m_{\mu\alpha} X^\alpha. \tag{2.0.8}$$

This is called "lowering the indices of X with m."

Similarly, given a covector with components Y_μ, we can use (m^{-1}) to form a vector Y^μ by raising the indices:

$$Y^\mu \overset{\text{def}}{=} \sum_{\alpha=0}^{3} (m^{-1})^{\mu\alpha} Y_\alpha. \tag{2.0.9}$$

These notions of duality are called *metric duality.* They are related to, but distinct from (roughly speaking by a minus sign in the first component), the notion of *basis duality* commonly introduced in linear algebra.

We will make use of *Einstein's summation convention*, in which we avoid writing many of the summation signs Σ to reduce the notational clutter. In particular, repeated indices, with one up and one down, are summed over their ranges. Here is an example:

$$X_\alpha Y^\alpha \overset{\text{def}}{=} \sum_{\alpha=0}^{3} X_\alpha Y^\alpha \overset{\text{def}}{=} \sum_{\alpha=0}^{3} X_\alpha Y^\alpha = m_{\alpha\beta} X^\beta Y^\alpha \overset{\text{def}}{=} m_{\alpha\beta} X^\beta Y^\beta = m_{\alpha\beta} X^\alpha Y^\beta, \tag{2.0.10}$$

where the last equality is a consequence of the symmetry property of m.

We now make the following important observation: the linear wave equation $-\partial_t^2 \phi + \Delta\phi = 0$ can be written as

$$(m^{-1})^{\alpha\beta} \partial_\alpha \partial_\beta \phi = 0. \tag{2.0.11}$$

We will return to this observation in a bit.

We first provide a standard division of vectors into three classes timelike, spacelike, null.

Definition 2.0.1.

(1) *Timelike* vectors: $m(X,X) \overset{\text{def}}{=} m_{\alpha\beta} X^\alpha X^\beta < 0$
(2) *Spacelike* vectors: $m(X,X) > 0$
(3) *Null* vectors: $m(X,X) = 0$
(4) *Causal* vectors: {Timelike vectors}$\cup$ {Null vectors}

We also will need to know when a vector is pointing "towards the future." This idea is captured by the next definition.

Definition 2.0.2. A vector $X \in \mathbb{R}^n$ is said to be *future-directed* if $X^0 > 0$.

2.1. Lorentz transformations.

Lorentz transformations play a very important role in the study of the linear wave equation.

Definition 2.1.1. A Lorentz transformation is a linear transformation $\Lambda^\mu{}_\nu$ (i.e., a matrix) that preserves the form of the Minkowski metric $m_{\mu\nu} \overset{\text{def}}{=} \text{diag}(-1,1,1,\cdots,1):$

$$\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}m_{\alpha\beta} = m_{\mu\nu}. \tag{2.1.1}$$

In standard matrix notation, (2.1.1) reads

$$\Lambda^T m\Lambda = m, \tag{2.1.2}$$

where T denotes the transpose.

By taking the determinant of each side of (2.1.2) and using the basic properties of the determinant, we see that $|\det(\Lambda)| = 1$. If $\det(\Lambda) = 1$, then Λ is said to be *proper* or *orientation preserving*.

It is easy to see that (2.1.1) is equivalent to

$$m(\Lambda X, \Lambda Y) = m(X, Y), \qquad \forall \text{ vectors } X, Y \in \mathbb{R}^{1+n}, \tag{2.1.3}$$

i.e., that the linear transformation Λ preserves the Minkowskian inner product. In (2.1.3), $m(X,Y) \overset{\text{def}}{=} m_{\alpha\beta}X^{\alpha}Y^{\beta}$ and ΛX is the vector with components $(\Lambda X)^{\mu} = \Lambda^{\mu}{}_{\alpha}X^{\alpha}$.

Also note that the left-hand side of (2.1.2) is connected to the linear-algebraic notion of *change of basis* on $\mathbb{R}^{1+n}$. More precisely, an important way of thinking about Lorentz transformations Λ is the following: if we have a standard rectangular coordinate system $(x^0, \cdots, x^n)$ on $\mathbb{R}^{1+n}$, and we change coordinates by defining $y^{\mu} \overset{\text{def}}{=} \Lambda^{\mu}{}_{\alpha}x^{\alpha}$, then relative to the new coordinate system $(y^0, \cdots, y^n)$, the Minkowski metric still has the same form $m_{\mu\nu} = \operatorname{diag}(-1, 1, 1, \cdots, 1)$. This statement would be false if, for example, we changed to polar spatial coordinates, or we dilated spacetime coordinates by setting $(y^0, \cdots, y^n) = \alpha(x^0, \cdots, x^n)$ for some constant $\alpha > 0$. Thus, the Lorentz transformations capture some *invariance properties* of m under certain special linear coordinate transformations.

Corollary 2.1.1. *If X is timelike, and Λ is a Lorentz transformation, then ΛX is also timelike. Analogous results also hold if X is spacelike or null.*

Proof. Corollary 2.1.1 easily follows from Definition 2.0.1 and (2.1.3). □

It can be checked that the Lorentz transformations form a group. In particular:

- If Λ is a Lorentz transformation, then so is Λ^{-1}.
- If Λ and Υ are Lorentz transformations, then so is their matrix product $\Lambda\Upsilon$, which has components $(\Lambda\Upsilon)^{\mu}{}_{\nu} \overset{\text{def}}{=} \Lambda^{\mu}{}_{\alpha}\Upsilon^{\alpha}{}_{\nu}$.

The condition (2.1.2) can be viewed as $(n+1)^2$ scalar equations. However, by the symmetry of m, there are plenty of redundancies, so that only $\frac{1}{2}(n+1)(n+2)$ of the equations are independent. This leaves $(n+1)^2 - \frac{1}{2}(n+1)(n+2) = \frac{1}{2}n(n+1)$ "free parameters" that determine the matrix Λ. Thus, the Lorentz transformations form a "$\frac{1}{2}n(n+1)$ dimensional" group.

It can be shown that the proper Lorentz group is *generated*[1] by the $\frac{(n)(n-1)}{2}$ dimensional subgroup of *spatial rotations*, and the n dimensional subgroup of proper *Lorentz boosts*. For the sake of concreteness let's focus on the physical case of $n = 3$ spatial dimensions.

Then the rotations about the x^3 axis are the set of linear transformations of the form

[1]By "generated," we mean that all proper Lorentz transformations can be built out of a finite number of products of boosts and spatial rotations.

$$\Lambda^{\alpha}{}_{\mu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.1.4}$$

where $\theta \in [0, 2\pi)$ is the counter-clockwise angle of rotation. Analogous matrices yield the rotations about the x^1 and x^2 axes. Note that the X^0 (i.e. "time") coordinate of vectors X is not affected by such transformations.

The (proper) Lorentz boosts are the famous linear transformations that play a distinguished role in Einstein's theory of special relativity. They are sometimes called *spacetime rotations*, because they intermix the time component X^0 of vectors X with their spatial components $X^1, X^2, \cdots, X^n$. The Lorentz boosts in the x^1 direction can be expressed as

$$\Lambda^{\alpha}{}_{\mu} = \begin{bmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.1.5}$$

where $\zeta \in (-\infty, \infty)$. Equivalently, (2.1.5) may be parameterized by

$$\Lambda^{\alpha}{}_{\mu} = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.1.6}$$

where $v \in (-1, 1)$ is a "velocity" and $\gamma = \sqrt{\frac{1}{1-v^2}}$. The requirement that $|v| < 1$ is directly connected to the idea that in special relativity, material particles should never "exceed the speed of light."

2.2. **Null frames.** It is often the case that the standard basis on $\mathbb{R}^{1+n}$ is not the best basis for analyzing solutions to the linear wave equation. One of the most useful bases is called a *null frame*, which can vary from spacetime point to spacetime point.

Definition 2.2.1. A null frame is a basis for $\mathbb{R}^{1+n}$ consisting of vectors $\{L, \underline{L}, e_{(1)}, \cdots, e_{(n-1)}\}$. Here, L and $\underline{L}$ are null vectors normalized by $m(L, \underline{L}) = -2$, and the $e_{(i)}$ are orthonormal vectors that span the m–orthogonal complement of span$(L, \underline{L})$: $m(e_{(i)}, e_{(j)}) = \delta_{ij}$, $m(L, e_{(i)}) = m(\underline{L}, e_{(i)}) = 0$, for $1 \leq i \leq j \leq n-1$. Note that the $e_{(i)}$ must form a basis for this complement; i.e., since they are m–orthonormal, they must be linearly independent.

In particular, we have the decomposition

$$\mathbb{R}^{1+n} = \mathrm{span}(L, \underline{L}) \oplus \mathrm{span}(e_{(1)}, \cdots, e_{(n-1)}), \tag{2.2.1}$$

where each of the two subspaces in the above direct sum are m–orthogonal.

Example 2.2.1. A common choice of a null frame is to take $L^{\mu} = (1, \omega^1, \cdots, \omega^n)$, $\underline{L}^{\mu} = (1, -\omega^1, \cdots, -\omega^n)$, and to take the $e_{(i)}$ to be any m–orthonormal basis for the m–orthogonal complement of span$(L, \underline{L})$. Note that this $n-1$ dimensional complementary space is spanned by the n *non-linearly independent* vectors $v^{\mu}_{(i)} \overset{\text{def}}{=} (0, -\omega^1, -\omega^2, \cdots, -\omega^{i-1}, 1-\omega^i, -\omega^{i+1}, \cdots, -\omega^n)$, $1 \leq i \leq n$. Here, $\omega^i \overset{\text{def}}{=} \frac{x^i}{r}$, and $r \overset{\text{def}}{=} \sqrt{\sum_{i=1}^n (x^i)^2}$ is the standard radial coordinate. Observe that $v_{(i)}$ is formed by subtracting of the "radial part" $(0, \omega_1, \cdots, \omega_n)$ from the standard spatial unit basis vector $b_{(i)}^{\mu} \overset{\text{def}}{=} (0, 0, \cdots, 0, \underbrace{1}_{i^{th}\text{ spatial slot}}, 0, \cdots, 0)$. Note that $\sum_{i=1}^n (\omega^i)^2 = 1$.

For this null frame, in terms of differential operators, $\nabla_L = \partial_t + \partial_r$, while $\nabla_{\underline{L}} = \partial_t - \partial_r$. The $\nabla_{e_{(i)}}$ are the *angular derivatives*, i.e., derivatives in directions tangential to the Euclidean spheres $S_{r,t} \overset{\text{def}}{=} \{(\tau, x^1, \cdots, x^n) \mid \tau = t, \sqrt{\sum_{i=1}^n (x^i)^2} = r.\}$

The following proposition shows that the Minkowski metric has a very nice form when expressed relative to a null frame.

Proposition 2.2.1 (Null frame decomposition of m). *If $\{L, \underline{L}, e_{(1)}, \cdots, e_{(n-1)}\}$ is a null frame, then we can decompose*

$$m_{\mu\nu} = -\frac{1}{2} L_{\mu} \underline{L}_{\nu} - \frac{1}{2} \underline{L}_{\mu} L_{\nu} + \not{m}_{\mu\nu}, \tag{2.2.2}$$

where $\not{m}_{\mu\nu}$ is positive-definite on the $m-$orthogonal complement of span$(L, \underline{L})$, and $\not{m}_{\mu\nu}$ vanishes on span$(L, \underline{L})$.

Similarly, by raising each index on both sides of (2.2.2) with m^{-1}, we have that

$$(m^{-1})^{\mu\nu} = -\frac{1}{2} L^{\mu} \underline{L}^{\nu} - \frac{1}{2} \underline{L}^{\mu} L^{\nu} + \not{m}^{\mu\nu}. \tag{2.2.3}$$

Proof. We define $\not{m}_{\mu\nu} \overset{\text{def}}{=} m_{\mu\nu} + \frac{1}{2} L_{\mu} \underline{L}_{\nu} + \frac{1}{2} \underline{L}_{\mu} L_{\nu}$. Since $m(L, L) = m(\underline{L}, \underline{L}) = 0$, and $m(L, \underline{L}) = -2$, it easily follows that $\not{m}(L, L) = \not{m}(L, \underline{L}) = \not{m}(\underline{L}, \underline{L}) = 0$. Thus, $\not{m}_{\mu\nu}$ vanishes on span$(L, \underline{L})$.

Since $m(L, e_{(i)}) = m(\underline{L}, e_{(i)}) = 0$ for $1 \le i \le n$, it easily follows that $\not{m}(L, e_{(i)}) = \not{m}(\underline{L}, e_{(i)}) = 0$.

Finally, it also easily follows that $\not{m}(e_{(i)}, e_{(j)}) = m(e_{(i)}, e_{(j)}) = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \ne j$, so that $\{e_{(i)}\}_{i=1}^{n-1}$ is an $\not{m}$ −orthonormal basis for the $m-$orthogonal complement of span$(L, \underline{L})$.

□

Remark 2.2.1. If the null frame is the one described in Example 2.2.1, then $\not{m}_{\mu\nu}$ is a metric that is positive definite in the "angular" directions, and 0 otherwise. In fact, $\not{m}$ is the standard Euclidean metric on the family Euclidean spheres $S_{r,t}$. $\not{m}$ is known as the *first fundamental form* of the spheres relative to m.

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