17. Problems

Problem 1. Prove that u_+, defined by (1.10) is linear.

Problem 2. Prove Lemma 1.8.

Hint(s). All functions here are supposed to be continuous, I just don't bother to keep on saying it.

(1) Recall, or check, that the local compactness of a metric space X means that for each point $x \in X$ there is an $\epsilon > 0$ such that the ball $\{y \in X; d(x, y) \leq \delta\}$ is compact for $\delta \leq \epsilon$.

(2) First do the case $n = 1$, so $K \Subset U$ is a compact set in an open subset.
   - (a) Given $\delta > 0$, use the local compactness of X, to cover K with a finite number of compact closed balls of radius at most δ.
   - (b) Deduce that if $\epsilon > 0$ is small enough then the set $\{x \in X; d(x, K) \leq \epsilon\}$, where
   $$d(x, K) = \inf_{y \in K} d(x, y),$$
   is compact.
   - (c) Show that $d(x, K)$, for K compact, is continuous.
   - (d) Given $\epsilon > 0$ show that there is a continuous function $g_\epsilon : \mathbb{R} \longrightarrow [0, 1]$ such that $g_\epsilon(t) = 1$ for $t \leq \epsilon/2$ and $g_\epsilon(t) = 0$ for $t > 3\epsilon/4$.
   - (e) Show that $f = g_\epsilon \circ d(\cdot, K)$ satisfies the conditions for $n = 1$ if $\epsilon > 0$ is small enough.

(3) Prove the general case by induction over n.
   - (a) In the general case, set $K' = K \cap U_1^{\complement}$ and show that the inductive hypothesis applies to K' and the U_j for $j > 1$; let f'_j, $j = 2, \ldots, n$ be the functions supplied by the inductive assumption and put $f' = \sum_{j \geq 2} f'_j$.
   - (b) Show that $K_1 = K \cap \{f' \leq \frac{1}{2}\}$ is a compact subset of U_1.
   - (c) Using the case $n = 1$ construct a function F for K_1 and U_1.
   - (d) Use the case $n = 1$ again to find G such that $G = 1$ on K and $\mathrm{supp}(G) \Subset \{f' + F > \frac{1}{2}\}$.
   - (e) Make sense of the functions
   $$f_1 = F \frac{G}{f' + F}, \; f_j = f'_j \frac{G}{f' + F}, \; j \geq 2$$
   and show that they satisfies the inductive assumptions.

Problem 3. Show that σ-algebras are closed under countable intersections.

Problem 4. (Easy) Show that if μ is a complete measure and $E \subset F$ where F is measurable and has measure 0 then $\mu(E) = 0$.

Problem 5. Show that compact subsets are measurable for any Borel measure. (This just means that compact sets are Borel sets if you follow through the tortuous terminology.)

Problem 6. Show that the smallest σ-algebra containing the sets

$$(a, \infty] \subset [-\infty, \infty]$$

for all $a \in \mathbb{R}$, generates what is called above the 'Borel' σ-algebra on $[-\infty, \infty]$.

Problem 7. Write down a careful proof of Proposition 1.1.

Problem 8. Write down a careful proof of Proposition 1.2.

Problem 9. Let X be the metric space

$$X = \{0\} \cup \{1/n; n \in \mathbb{N} = \{1, 2, \ldots\}\} \subset \mathbb{R}$$

with the induced metric (i.e. the same distance as on $\mathbb{R}$). Recall why X is compact. Show that the space $\mathcal{C}_0(X)$ and its dual are infinite dimensional. Try to describe the dual space in terms of sequences; at least *guess* the answer.

Problem 10. For the space $Y = \mathbb{N} = \{1, 2, \ldots\} \subset \mathbb{R}$, describe $\mathcal{C}_0(Y)$ and guess a description of its dual in terms of sequences.

Problem 11. Let $(X, \mathcal{M}, \mu)$ be any measure space (so μ is a measure on the σ-algebra $\mathcal{M}$ of subsets of X). Show that the set of equivalence classes of μ-integrable functions on X, with the equivalence relation given by (4.8), is a normed linear space with the usual linear structure and the norm given by

$$\|f\| = \int_X |f| d\mu.$$

Problem 12. Let $(X, \mathcal{M})$ be a set with a σ-algebra. Let $\mu : \mathcal{M} \to \mathbb{R}$ be a finite measure in the sense that $\mu(\phi) = 0$ and for any $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$ with $E_i \cap E_j = \phi$ for $i \neq j$,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \tag{17.1}$$

with the series on the right *always* absolutely convergenct (i.e., this is part of the requirement on μ). Define

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)| \tag{17.2}$$

for $E \in \mathcal{M}$, with the supremum over *all* measurable decompositions $E = \bigcup_{i=1}^{\infty} E_i$ with the E_i disjoint. Show that $|\mu|$ is a finite, positive measure.

Hint 1. You must show that $|\mu|(E) = \sum_{i=1}^{\infty} |\mu|(A_i)$ if $\bigcup_i A_i = E$, $A_i \in \mathcal{M}$ being disjoint. Observe that if $A_j = \bigcup_l A_{jl}$ is a measurable decomposition of A_j then together the A_{jl} give a decomposition of E. Similarly, if $E = \bigcup_j E_j$ is any such decomposition of E then $A_{jl} = A_j \cap E_l$ gives such a decomposition of A_j.

Hint 2. See [5] p. 117!

Problem 13. (Hahn Decomposition)

With assumptions as in Problem 12:

(1) Show that $\mu_+ = \frac{1}{2}(|\mu| + \mu)$ and $\mu_- = \frac{1}{2}(|\mu| - \mu)$ are positive measures, $\mu = \mu_+ - \mu_-$. Conclude that the definition of a measure based on (4.16) is the *same* as that in Problem 12.

(2) Show that $\mu_\pm$ so constructed are orthogonal in the sense that there is a set $E \in \mathcal{M}$ such that $\mu_-(E) = 0$, $\mu_+(X \setminus E) = 0$.

Hint. Use the definition of $|\mu|$ to show that for any $F \in \mathcal{M}$ and any $\epsilon > 0$ there is a subset $F' \in \mathcal{M}$, $F' \subset F$ such that $\mu_+(F') \geq \mu_+(F) - \epsilon$ and $\mu_-(F') \leq \epsilon$. Given $\delta > 0$ apply this result repeatedly (say with $\epsilon = 2^{-n}\delta$) to find a decreasing sequence of sets $F_1 = X$, $F_n \in \mathcal{M}$, $F_{n+1} \subset F_n$ such that $\mu_+(F_n) \geq \mu_+(F_{n-1}) - 2^{-n}\delta$ and $\mu_-(F_n) \leq 2^{-n}\delta$. Conclude that $G = \bigcap_n F_n$ has $\mu_+(G) \geq \mu_+(X) - \delta$ and $\mu_-(G) = 0$. Now let G_m be chosen this way with $\delta = 1/m$. Show that $E = \bigcup_m G_m$ is as required.

Problem 14. Now suppose that μ is a finite, positive Radon measure on a locally compact metric space X (meaning a finite positive Borel measure outer regular on Borel sets and inner regular on open sets). Show that μ is inner regular on all Borel sets and hence, given $\epsilon > 0$ and $E \in \mathcal{B}(X)$ there exist sets $K \subset E \subset U$ with K compact and U open such that $\mu(K) \geq \mu(E) - \epsilon$, $\mu(E) \geq \mu(U) - \epsilon$.

Hint. First take U open, then use *its* inner regularity to find K with $K' \Subset U$ and $\mu(K') \geq \mu(U) - \epsilon/2$. How big is $\mu(E \backslash K')$? Find $V \supset K' \backslash E$ with V open and look at $K = K' \backslash V$.

Problem 15. Using Problem 14 show that if μ is a finite Borel measure on a locally compact metric space X then the following three conditions are equivalent

(1) $\mu = \mu_1 - \mu_2$ with μ_1 and μ_2 both positive finite Radon measures.
(2) $|\mu|$ is a finite positive Radon measure.
(3) μ_+ and μ_- are finite positive Radon measures.

Problem 16. Let $\| \ \|$ be a norm on a vector space V. Show that $\|u\| = (u,u)^{1/2}$ for an inner product satisfying (5.1) - (5.4) if and only if the parallelogram law holds for every pair $u, v \in V$.

Hint (From Dimitri Kountourogiannis)

If $\| \cdot \|$ comes from an inner product, then it must satisfy the polarisation identity:

$$(x,y) = 1/4(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 - i\|x-iy\|^2)$$

i.e, the inner product is recoverable from the norm, so use the RHS (right hand side) to define an inner product on the vector space. You will need the paralellogram law to verify the additivity of the RHS. Note the polarization identity is a bit more transparent for real vector spaces. There we have

$$(x,y) = 1/2(\|x+y\|^2 - \|x-y\|^2)$$

both are easy to prove using $\|a\|^2 = (a,a)$.

Problem 17. Show (Rudin does it) that if $u : \mathbb{R}^n \to \mathbb{C}$ has continuous partial derivatives then it is differentiable at each point in the sense of (6.5).

Problem 18. Consider the function $f(x) = \langle x \rangle^{-1} = (1+|x|^2)^{-1/2}$. Show that

$$\frac{\partial f}{\partial x_j} = l_j(x) \cdot \langle x \rangle^{-3}$$

with $l_j(x)$ a linear function. Conclude by *induction* that $\langle x \rangle^{-1} \in \mathcal{C}_0^k(\mathbb{R}^n)$ for *all* k.

Problem 19. Show that $\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n)$.

Problem 20. Prove (7.7), probably by induction over k.

Problem 21. Prove Lemma 7.4.

Hint. Show that a set $U \ni 0$ in $\mathcal{S}(\mathbb{R}^n)$ is a neighbourhood of 0 if and only if for some k and $\epsilon > 0$ it contains a set of the form

$$\left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) \, ; \sum_{\substack{|\alpha| \le k, \\ |\beta| \le k}} \sup \left| x^\alpha D^\beta \varphi \right| < \epsilon \right\} .$$

Problem 22. Prove (8.7), by estimating the integrals.

Problem 23. Prove (8.9) where

$$\psi_j(z; x') = \int_0^{\prime} \frac{\partial \psi}{\partial z_j}(z + tx') \, dt \, .$$

Problem 24. Prove (8.20). You will probably have to go back to first principles to do this. Show that it is enough to assume $u \ge 0$ has compact support. Then show it is enough to assume that u is a simple, and integrable, function. Finally look at the definition of Lebesgue measure and show that if $E \subset \mathbb{R}^n$ is Borel and has finite Lebesgue measure then

$$\lim_{|t| \to \infty} \mu(E \backslash (E + t)) = 0$$

where $\mu =$ Lebesgue measure and

$$E + t = \{ p \in \mathbb{R}^n \, ; \, p' + t \, , \, p' \in E \} \, .$$

Problem 25. Prove Leibniz' formula

$$D^\alpha{}_x(\varphi\psi) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\alpha{}_x \varphi \cdot d_x^{\alpha - \beta} \psi$$

for any $\mathcal{C}^\infty$ functions and φ and ψ. Here α and β are multiindices, $\beta \le \alpha$ means $\beta_j \le \alpha_j$ for each $j_?$ and

$$\binom{\alpha}{\beta} = \prod_j \binom{\alpha_j}{\beta_j} .$$

I suggest induction!

Problem 26. Prove the generalization of Proposition 8.10 that $u \in \mathcal{S}'(\mathbb{R}^n)$, $\mathrm{supp}(w) \subset \{0\}$ implies there are constants $c\alpha$, $|\alpha| \le m$, for some m, such that

$$u = \sum_{|\alpha| \le m} c_\alpha D^\alpha \delta \, .$$

Hint This is not so easy! I would be happy if you can show that $u \in M(\mathbb{R}^n)$, $\operatorname{supp} u \subset \{0\}$ implies $u = c\delta$. To see this, you can show that

$$\begin{aligned} &\varphi \in \mathcal{S}(\mathbb{R}^n),\ \varphi(0) = 0 \\ &\qquad\qquad \Rightarrow \exists \varphi_j \in \mathcal{S}(\mathbb{R}^n)\,,\ \varphi_j(x) = 0 \text{ in } |x| \le \epsilon_j > 0(\downarrow 0)\,, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \sup|\varphi_j - \varphi| \to 0 \text{ as } j \to \infty\,. \end{aligned}$$

To prove the general case you need something similar — that given m, if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $D^\alpha{}_x\varphi(0) = 0$ for $|\alpha| \le m$ then $\exists\, \varphi_j \in \mathcal{S}(\mathbb{R}^n)$, $\varphi_j = 0$ in $|x| \le \epsilon_j$, $\epsilon_j \downarrow 0$ such that $\varphi_j \to \varphi$ in the $\mathcal{C}^m$ norm.

Problem 27. If $m \in \mathbb{N}$, $m' > 0$ show that $u \in H^m(\mathbb{R}^n)$ and $D^\alpha u \in H^{m'}(\mathbb{R}^n)$ for all $|\alpha| \le m$ implies $u \in H^{m+m'}(\mathbb{R}^n)$. Is the converse true?

Problem 28. Show that every element $u \in L^2(\mathbb{R}^n)$ can be written as a sum

$$u = u_0 + \sum_{j=1}^n D_j u_j\,,\ u_j \in H^1(\mathbb{R}^n)\,,\ j = 0, \ldots, n\,.$$

Problem 29. Consider for $n = 1$, the locally integrable function (the Heaviside function),

$$H(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 1\,. \end{cases}$$

Show that $D_x H(x) = c\delta$; what is the constant c?

Problem 30. For what range of orders m is it true that $\delta \in H^m(\mathbb{R}^n)\,,\ \delta(\varphi) = \varphi(0)$?

Problem 31. Try to write the Dirac measure explicitly (as possible) in the form (10.8). How many derivatives do you think are necessary?

Problem 32. Go through the computation of $\overline{\partial}E$ again, but cutting out a disk $\{x^2 + y^2 \le \epsilon^2\}$ instead.

Problem 33. Consider the Laplacian, (11.4), for $n = 3$. Show that $E = c(x^2 + y^2)^{-1/2}$ is a fundamental solution for some value of c.

Problem 34. Recall that a topology on a set X is a collection $\mathcal{F}$ of subsets (called the *open* sets) with the properties, $\phi \in \mathcal{F}$, $X \in \mathcal{F}$ and $\mathcal{F}$ is closed under finite intersections and arbitrary unions. Show that the following definition of an open set $U \subset \mathcal{S}'(\mathbb{R}^n)$ defines a topology:

$$\begin{aligned} &\forall\ u \in U \text{ and all } \varphi \in \mathcal{S}(\mathbb{R}^n)\ \exists \epsilon > 0 \text{ st.} \\ &\qquad\qquad |(u' - u)(\varphi)| < \epsilon \Rightarrow u' \in U\,. \end{aligned}$$

This is called the weak topology (because there are very few open sets). Show that $u_j \to u$ weakly in $\mathcal{S}'(\mathbb{R}^n)$ means that for every open set $U \ni u$ $\exists N$ st. $u_j \in U \ \forall\ j \geq N$.

Problem 35. Prove (11.18) where $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$.

Problem 36. Show that for fixed $v \in \mathcal{S}'(\mathbb{R}^n)$ with compact support

$$\mathcal{S}(\mathbb{R}^n) \ni \varphi \longmapsto v * \varphi \in \mathcal{S}(\mathbb{R}^n)$$

is a continuous linear map.

Problem 37. Prove the ?? to properties in Theorem 11.6 for $u * v$ where $u \in \mathcal{S}'(\mathbb{R}^n)$ and $v \in \mathcal{S}'(\mathbb{R}^n)$ with at least one of them having compact support.

Problem 38. Use Theorem 11.9 to show that if $P(D)$ is hypoelliptic then *every* parametrix $F \in \mathcal{S}(\mathbb{R}^n)$ has $\operatorname{sing\,supp}(F) = \{0\}$.

Problem 39. Show that if $P(D)$ is an ellipitic differential operator of order m, $u \in L^2(\mathbb{R}^n)$ and $P(D)u \in L^2(\mathbb{R}^n)$ then $u \in H^m(\mathbb{R}^n)$.

Problem 40 (Taylor's theorem). . Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real-valued function which is k times continuously differentiable. Prove that there is a polynomial p and a continuous function v such that

$$u(x) = p(x) + v(x) \text{ where } \lim_{|x|\downarrow 0} \frac{|v(x)|}{|x|^k} = 0.$$

Problem 41. Let $\mathcal{C}(\mathbb{B}^n)$ be the space of continuous functions on the (closed) unit ball, $\mathbb{B}^n = \{x \in \mathbb{R}^n; |x| \leq 1\}$. Let $\mathcal{C}_0(\mathbb{B}^n) \subset \mathcal{C}(\mathbb{B}^n)$ be the subspace of functions which vanish at each point of the boundary and let $\mathcal{C}(\mathbb{S}^{n-1})$ be the space of continuous functions on the unit sphere. Show that inclusion and restriction to the boundary gives a short exact sequence

$$\mathcal{C}_0(\mathbb{B}^n) \hookrightarrow \mathcal{C}(\mathbb{B}^n) \longrightarrow \mathcal{C}(\mathbb{S}^{n-1})$$

(meaning the first map is injective, the second is surjective and the image of the first is the null space of the second.)

Problem 42 (Measures). A measure on the ball is a continuous linear functional $\mu : \mathcal{C}(\mathbb{B}^n) \longrightarrow \mathbb{R}$ where continuity is with respect to the supremum norm, i.e. there must be a constant C such that

$$|\mu(f)| \leq C \sup_{x \in \mathbb{R}^n} |f(x)| \ \forall\ f \in \mathcal{C}(\mathbb{B}^n).$$

Let $M(\mathbb{B}^n)$ be the linear space of such measures. The space $M(\mathbb{S}^{n-1})$ of measures on the sphere is defined similarly. Describe an injective map

$$M(\mathbb{S}^{n-1}) \longrightarrow M(\mathbb{B}^n).$$

Can you define another space so that this can be extended to a short exact sequence?

Problem 43. Show that the Riemann integral defines a measure

$$\mathcal{C}(\mathbb{B}^n) \ni f \longmapsto \int_{\mathbb{B}^n} f(x)dx. \tag{17.3}$$

Problem 44. If $g \in \mathcal{C}(\mathbb{B}^n)$ and $\mu \in M(\mathbb{B}^n)$ show that $g\mu \in M(\mathbb{B}^n)$ where $(g\mu)(f) = \mu(fg)$ for all $f \in \mathcal{C}(\mathbb{B}^n)$. Describe all the measures with the property that

$$x_j\mu = 0 \text{ in } M(\mathbb{B}^n) \text{ for } j = 1, \dots, n.$$

Problem 45 (Hörmander, Theorem 3.1.4). Let $I \subset \mathbb{R}$ be an open, non-empty interval.

i) Show (you may use results from class) that there exists $\psi \in \mathcal{C}^\infty_c(I)$ with $\int_{\mathbb{R}} \psi(x)ds = 1$.
ii) Show that any $\phi \in \mathcal{C}^\infty_c(I)$ may be written in the form
$$\phi = \tilde{\phi} + c\psi,\ c \in \mathbb{C},\ \tilde{\phi} \in \mathcal{C}^\infty_c(I) \text{ with } \int_{\mathbb{R}} \tilde{\phi} = 0.$$
iii) Show that if $\tilde{\phi} \in \mathcal{C}^\infty_c(I)$ and $\int_{\mathbb{R}} \tilde{\phi} = 0$ then there exists $\mu \in \mathcal{C}^\infty_c(I)$ such that $\frac{d\mu}{dx} = \tilde{\phi}$ in I.
iv) Suppose $u \in \mathcal{C}^{-\infty}(I)$ satisfies $\frac{du}{dx} = 0$, i.e.
$$u(-\frac{d\phi}{dx}) = 0 \ \forall \ \phi \in \mathcal{C}^\infty_c(I),$$
show that $u = c$ for some constant c.
v) Suppose that $u \in \mathcal{C}^{-\infty}(I)$ satisfies $\frac{du}{dx} = c$, for some constant c, show that $u = cx + d$ for some $d \in \mathbb{C}$.

Problem 46. [Hörmander Theorem 3.1.16]

i) Use Taylor's formula to show that there is a fixed $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ such that any $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ can be written in the form
$$\phi = c\psi + \sum_{j=1}^n x_j\psi_j$$
where $c \in \mathbb{C}$ and the $\psi_j \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ depend on ϕ.
ii) Recall that δ_0 is the distribution defined by
$$\delta_0(\phi) = \phi(0) \ \forall \ \phi \in \mathcal{C}^\infty_c(\mathbb{R}^n);$$
explain why $\delta_0 \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$.
iii) Show that if $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and $u(x_j\phi) = 0$ for all $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ and $j = 1, \dots, n$ then $u = c\delta_0$ for some $c \in \mathbb{C}$.

iv) Define the 'Heaviside function'

$$H(\phi) = \int_0^\infty \phi(x)dx \;\forall\; \phi \in \mathcal{C}_c^\infty(\mathbb{R});$$

show that $H \in \mathcal{C}^{-\infty}(\mathbb{R})$.

v) Compute $\frac{d}{dx}H \in \mathcal{C}^{-\infty}(\mathbb{R})$.

Problem 47. Using Problems 45 and 46, find all $u \in \mathcal{C}^{-\infty}(\mathbb{R})$ satisfying the differential equation

$$x\frac{du}{dx} = 0 \text{ in } \mathbb{R}.$$

These three problems are all about homogeneous distributions on the line, extending various things using the fact that

$$x_+^z = \begin{cases} \exp(z \log x) & x > 0 \\ 0 & x \le 0 \end{cases}$$

is a continuous function on $\mathbb{R}$ if $\operatorname{Re} z > 0$ and is differentiable if $\operatorname{Re} z > 1$ and then satisfies

$$\frac{d}{dx}x_+^z = zx_+^{z-1}.$$

We used this to define

$$x_+^z = \frac{1}{z+k}\frac{1}{z+k-1}\cdots\frac{1}{z+1}\frac{d^k}{dx^k}x_+^{z+k} \text{ if } z \in \mathbb{C}\setminus -\mathbb{N}. \tag{17.4}$$

Problem 48. [Hadamard regularization]

i) Show that (17.4) just means that for each $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$

$$x_+^z(\phi) = \frac{(-1)^k}{(z+k)\cdots(z+1)}\int_0^\infty \frac{d^k\phi}{dx^k}(x)x^{z+k}dx,\ \operatorname{Re} z > -k,\ z \notin -\mathbb{N}.$$

ii) Use integration by parts to show that

$$x_+^z(\phi) = \lim_{\epsilon\downarrow 0}\left[\int_\epsilon^\infty \phi(x)x^z dx - \sum_{j=1}^k C_j(\phi)\epsilon^{z+j}\right],\ \operatorname{Re} z > -k,\ z \notin -\mathbb{N} \tag{17.5}$$

for certain constants $C_j(\phi)$ which you should give explicitly. [This is called Hadamard regularization after Jacques Hadamard, feel free to look at his classic book [3].]

iii) Assuming that $-k+1 \ge \operatorname{Re} z > -k$, $z \ne -k+1$, show that there can only be one set of the constants with $j < k$ (for each choice of $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$) such that the limit in (17.5) exists.

iiv) Use ii), and maybe iii), to show that

$$\frac{d}{dx}x_+^z = zx_+^{z-1} \text{ in } \mathcal{C}^{-\infty}(\mathbb{R}) \ \forall\ z \notin -\mathbb{N}_0 = \{0, 1, \dots\}.$$

v) Similarly show that $xx_+^z = x_+^{z+1}$ for all $z \notin -\mathbb{N}$.
vi) Show that $x_+^z = 0$ in $x < 0$ for all $z \notin -\mathbb{N}$. (Duh.)

Problem 49. [Null space of $x\frac{d}{dx} - z$]

i) Show that if $u \in \mathcal{C}^{-\infty}(\mathbb{R})$ then $\tilde{u}(\phi) = u(\tilde{\phi})$, where $\tilde{\phi}(x) = \phi(-x)\ \forall\ \phi \in \mathcal{C}_c^\infty(\mathbb{R})$, defines an element of $\mathcal{C}^{-\infty}(\mathbb{R})$. What is $\tilde{u}$ if $u \in \mathcal{C}^0(\mathbb{R})$? Compute $\widetilde{\delta_0}$.
ii) Show that $\frac{d}{dx}\tilde{u} = -\widetilde{\frac{d}{dx}u}$.
iii) Define $x_-^z = \widetilde{x_+^z}$ for $z \notin -\mathbb{N}$ and show that $\frac{d}{dx}x_-^z = -zx_-^{z-1}$ and $xx_-^z = -x_-^{z+1}$.
iv) Suppose that $u \in \mathcal{C}^{-\infty}(\mathbb{R})$ satisfies the distributional equation $(x\frac{d}{dx} - z)u = 0$ (meaning of course, $x\frac{du}{dx} = zu$ where z is a constant). Show that

$$u\big|_{x>0} = c_+x_-^z\big|_{x>0} \text{ and } u\big|_{x<0} = c_-x_-^z\big|_{x<0}$$

for some constants $c_\pm$. Deduce that $v = u - c_+x_+^z - c_-x_-^z$ satisfies

$$(x\frac{d}{dx} - z)v = 0 \text{ and } \operatorname{supp}(v) \subset \{0\}. \tag{17.6}$$

v) Show that for each $k \in \mathbb{N}$, $(x\frac{d}{dx} + k + 1)\frac{d^k}{dx^k}\delta_0 = 0$.
vi) Using the *fact* that any $v \in \mathcal{C}^{-\infty}(\mathbb{R})$ with $\operatorname{supp}(v) \subset \{0\}$ is a finite sum of constant multiples of the $\frac{d^k}{dx^k}\delta_0$, show that, for $z \notin -\mathbb{N}$, the only solution of (17.6) is $v = 0$.
vii) Conclude that for $z \notin -\mathbb{N}$

$$\left\{u \in \mathcal{C}^{-\infty}(\mathbb{R}); (x\frac{d}{dx} - z)u = 0\right\} \tag{17.7}$$

is a two-dimensional vector space.

Problem 50. [Negative integral order] To do the same thing for negative integral order we need to work a little differently. Fix $k \in \mathbb{N}$.

i) We define *weak convergence* of distributions by saying $u_n \to u$ in $\mathcal{C}_c^\infty(X)$, where $u_n, u \in \mathcal{C}^{-\infty}(X)$, $X \subset \mathbb{R}^n$ being open, if $u_n(\phi) \to u(\phi)$ for each $\phi \in \mathcal{C}_c^\infty(X)$. Show that $u_n \to u$ implies that $\frac{\partial u_n}{\partial x_j} \to \frac{\partial u}{\partial x_j}$ for each $j = 1, \dots, n$ and $fu_n \to fu$ if $f \in \mathcal{C}^\infty(X)$.

ii) Show that $(z+k)x_+^z$ is weakly continuous as $z \to -k$ in the sense that for any sequence $z_n \to -k$, $z_n \notin -\mathbb{N}$, $(z_n+k)x_+^{z_n} \to v_k$ where

$$v_k = \frac{1}{-1} \cdots \frac{1}{-k+1} \frac{d^{k+1}}{dx^{k+1}} x_+, \ x_+ = x_+^1.$$

iii) Compute v_k, including the constant factor.
iv) Do the same thing for $(z+k)x_-^z$ as $z \to -k$.
v) Show that there is a linear combination $(k+z)(x_+^z + c(k)x_-^z)$ such that as $z \to -k$ the limit is zero.
vi) If you get this far, show that in fact $x_+^z + c(k)x_-^z$ also has a weak limit, u_k, as $z \to -k$. [This may be the hardest part.]
vii) Show that this limit distribution satisfies $(x\frac{d}{dx} + k)u_k = 0$.
viii) Conclude that (17.7) does in fact hold for $z \in -\mathbb{N}$ as well. [There are still some things to prove to get this.]

Problem 51. Show that for any set $G \subset \mathbb{R}^n$

$$v^*(G) = \inf \sum_{i=1}^{\infty} v(A_i)$$

where the infimum is taken over coverings of G by rectangular sets (products of intervals).

Problem 52. Show that a σ-algebra is closed under countable intersections.

Problem 53. Show that compact sets are Lebesgue measurable and have finite volume and also show the inner regularity of the Lebesgue measure on open sets, that is if E is open then

$$v(E) = \sup\{v(K); K \subset E, \ K \text{ compact}\}. \tag{17.8}$$

Problem 54. Show that a set $B \subset \mathbb{R}^n$ is Lebesgue measurable if and only if

$$v^*(E) = v^*(E \cap B) + v^*(E \cap B^{\complement}) \ \forall \text{ open } E \subset \mathbb{R}^n.$$

[The definition is this for all $E \subset \mathbb{R}^n$.]

Problem 55. Show that a real-valued continuous function $f : U \longrightarrow \mathbb{R}$ on an open set, is Lebesgue measurable, in the sense that $f^{-1}(I) \subset U \subset \mathbb{R}^n$ is measurable for each interval I.

Problem 56. Hilbert space and the Riesz representation theorem. If you need help with this, it can be found in lots of places – for instance [6] has a nice treatment.

i) A pre-Hilbert space is a vector space V (over $\mathbb{C}$) with a 'positive definite sesquilinear inner product' i.e. a function

$$V \times V \ni (v, w) \longmapsto \langle v, w\rangle \in \mathbb{C}$$

satisfying

- $\langle w, v\rangle = \overline{\langle v, w\rangle}$
- $\langle a_1v_1 + a_2v_2, w\rangle = a_1\langle v_1, w\rangle + a_2\langle v_2, w\rangle$
- $\langle v, v\rangle \geq 0$
- $\langle v, v\rangle = 0 \Rightarrow v = 0$.

Prove Schwarz' inequality, that

$$|\langle u, v\rangle| \leq \langle u\rangle^{\frac{1}{2}}\langle v\rangle^{\frac{1}{2}} \ \forall\ u, v \in V.$$

Hint: Reduce to the case $\langle v, v\rangle = 1$ and then expand

$$\langle u - \langle u, v\rangle v, u - \langle u, v\rangle v\rangle \geq 0.$$

ii) Show that $\|v\| = \langle v, v\rangle^{1/2}$ is a norm and that it satisfies the parallelogram law:

$$\|v_1 + v_2\|^2 + \|v_1 - v_2\|^2 = 2\|v_1\|^2 + 2\|v_2\|^2 \ \forall\ v_1, v_2 \in V. \tag{17.9}$$

iii) Conversely, suppose that V is a linear space over $\mathbb{C}$ with a norm which satisfies (17.9). Show that

$$4\langle v, w\rangle = \|v + w\|^2 - \|v - w\|^2 + i\|v + iw\|^2 - i\|v - iw\|^2$$

defines a pre-Hilbert inner product which gives the original norm.

iv) Let V be a Hilbert space, so as in (i) but complete as well. Let $C \subset V$ be a closed non-empty convex subset, meaning $v, w \in C \Rightarrow (v + w)/2 \in C$. Show that there exists a unique $v \in C$ minimizing the norm, i.e. such that

$$\|v\| = \inf_{w \in C} \|w\|.$$

Hint: Use the parallelogram law to show that a norm minimizing sequence is Cauchy.

v) Let $u : H \longrightarrow \mathbb{C}$ be a continuous linear functional on a Hilbert space, so $|u(\varphi)| \leq C\|\varphi\| \ \forall\ \varphi \in H$. Show that $N = \{\varphi \in H; u(\varphi) = 0\}$ is closed and that if $v_0 \in H$ has $u(v_0) \neq 0$ then each $v \in H$ can be written uniquely in the form

$$v = cv_0 + w,\ c \in \mathbb{C},\ w \in N.$$

vi) With u as in v), not the zero functional, show that there exists a unique $f \in H$ with $u(f) = 1$ and $\langle w, f\rangle = 0$ for all $w \in N$.

Hint: Apply iv) to $C = \{g \in V; u(g) = 1\}$.

vii) Prove the Riesz Representation theorem, that every continuous linear functional on a Hilbert space is of the form

$$u_f : H \ni \varphi \longmapsto \langle \varphi, f \rangle \text{ for a unique } f \in H.$$

Problem 57. Density of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$.

i) Recall in a few words why simple integrable functions are dense in $L^1(\mathbb{R}^n)$ with respect to the norm $\|f\|_{L^1} = \int_{\mathbb{R}^n} |f(x)|dx$.
ii) Show that simple functions $\sum_{j=1}^N c_j \chi(U_j)$ where the U_j are open and bounded are also dense in $L^1(\mathbb{R}^n)$.
iii) Show that if U is open and bounded then $F(y) = v(U \cap U_y)$, where $U_y = \{z \in \mathbb{R}^n; z = y + y', y' \in U\}$ is continuous in $y \in \mathbb{R}^n$ and that

$$v(U \cap U_y^{\complement}) + v(U^{\complement} \cap U_y) \to 0 \text{ as } y \to 0.$$

iv) If U is open and bounded and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ show that

$$f(x) = \int_U \varphi(x - y)dy \in \mathcal{C}_c^\infty(\mathbb{R}^n).$$

v) Show that if U is open and bounded then

$$\sup_{|y|\le\delta} \int |\chi_U(x) - \chi_U(x - y)|dx \to 0 \text{ as } \delta \downarrow 0.$$

vi) If U is open and bounded and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\varphi \ge 0$, $\int \varphi = 1$ then

$$f_\delta \to \chi_U \text{ in } L^1(\mathbb{R}^n) \text{ as } \delta \downarrow 0$$

where

$$f_\delta(x) = \delta^{-n} \int \varphi\left(\frac{y}{\delta}\right) \chi_U(x - y)dy.$$

Hint: Write $\chi_U(x) = \delta^{-n} \int \varphi\left(\frac{y}{\delta}\right) \chi_U(x)$ and use v).

vii) Conclude that $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.
viii) Show that $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for any $1 \le p < \infty$.

Problem 58. Schwartz representation theorem. Here we (well you) come to grips with the general structure of a tempered distribution.

i) Recall briefly the proof of the Sobolev embedding theorem and the corresponding estimate

$$\sup_{x\in\mathbb{R}^n} |\phi(x)| \le C\|\phi\|_{H^m}, \ \frac{n}{2} < m \in \mathbb{R}.$$

ii) For $m = n + 1$ write down a(n equivalent) norm on the right in a form that does not involve the Fourier transform.

iii) Show that for any $\alpha \in \mathbb{N}_0$

$$|D^\alpha\left((1+|x|^2)^N\phi\right)| \le C_{\alpha,N}\sum_{\beta\le\alpha}(1+|x|^2)^N|D^\beta\phi|.$$

iv) Deduce the general estimates

$$\sup_{\substack{|\alpha|\le N\\ x\in\mathbb{R}^n}}(1+|x|^2)^N|D^\alpha\phi(x)| \le C_N\|(1+|x|^2)^N\phi\|_{H^{N+n+1}}.$$

v) Conclude that for each tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ there is an integer N and a constant C such that

$$|u(\phi)| \le C\|(1+|x|^2)^N\phi\|_{H^{2N}}\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n).$$

vi) Show that $v = (1+|x|^2)^{-N}u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies

$$|v(\phi)| \le C\|(1+|D|^2)^N\phi\|_{L^2}\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n).$$

vi) Recall (from class or just show it) that if v is a tempered distribution then there is a unique $w \in \mathcal{S}'(\mathbb{R}^n)$ such that $(1+|D|^2)^Nw = v$.

vii) Use the Riesz Representation Theorem to conclude that for each tempered distribution u there exists N and $w \in L^2(\mathbb{R}^n)$ such that

$$u = (1+|D|^2)^N(1+|x|^2)^Nw. \tag{17.10}$$

viii) Use the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ (and the fact that it is an isomorphism on $L^2(\mathbb{R}^n)$) to show that any tempered distribution can be written in the form

$$u = (1+|x|^2)^N(1+|D|^2)^Nw \text{ for some } N \text{ and some } w \in L^2(\mathbb{R}^n).$$

ix) Show that any tempered distribution can be written in the form

$$u = (1+|x|^2)^N(1+|D|^2)^{N+n+1}\tilde{w} \text{ for some } N \text{ and some } \tilde{w} \in H^{2(n+1)}(\mathbb{R}^n).$$

x) Conclude that any tempered distribution can be written in the form

$$u = (1+|x|^2)^N(1+|D|^2)^MU \text{ for some } N, M \text{ and a bounded continuous function } U$$

Problem 59. Distributions of compact support.

i) Recall the definition of the support of a distribution, defined in terms of its complement

$$\mathbb{R}^n\backslash\operatorname{supp}(u) = \left\{p \in \mathbb{R}^n; \exists\ U \subset \mathbb{R}^n,\ \text{open, with } p \in U \text{ such that } u\big|_U = 0\right\}$$

ii) Show that if $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ satisfy

$$\operatorname{supp}(u) \cap \operatorname{supp}(\phi) = \emptyset$$

then $u(\phi) = 0$.

iii) Consider the space $\mathcal{C}^\infty(\mathbb{R}^n)$ of all smooth functions on $\mathbb{R}^n$, without restriction on supports. Show that for each N

$$\|f\|_{(N)} = \sup_{|\alpha| \le N,\ |x| \le N} |D^\alpha f(x)|$$

is a seminorn on $\mathcal{C}^\infty(\mathbb{R}^n)$ (meaning it satisfies $\|f\| \ge 0$, $\|cf\| = |c|\|f\|$ for $c \in \mathbb{C}$ and the triangle inequality but that $\|f\| = 0$ does not necessarily imply that $f = 0$.)

iv) Show that $\mathcal{C}_c^\infty(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)$ is dense in the sense that for each $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ there is a sequence f_n in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\|f - f_n\|_{(N)} \to 0$ for each N.

v) Let $\mathcal{E}'(\mathbb{R}^n)$ temporarily (or permanantly if you prefer) denote the dual space of $\mathcal{C}^\infty(\mathbb{R}^n)$ (which is also written $\mathcal{E}(\mathbb{R}^n)$), that is, $v \in \mathcal{E}'(\mathbb{R}^n)$ is a linear map $v : \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathbb{C}$ which is continuous in the sense that for some N

$$|v(f)| \le C\|f\|_{(N)} \ \forall\ f \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{17.11}$$

Show that such a v 'is' a distribution and that the map $\mathcal{E}'(\mathbb{R}^n) \longrightarrow \mathcal{C}^{-\infty}(\mathbb{R}^n)$ is injective.

vi) Show that if $v \in \mathcal{E}'(\mathbb{R}^n)$ satisfies (17.11) and $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ has $f = 0$ in $|x| < N + \epsilon$ for some $\epsilon > 0$ then $v(f) = 0$.

vii) Conclude that each element of $\mathcal{E}'(\mathbb{R}^n)$ has compact support when considered as an element of $\mathcal{C}^{-\infty}(\mathbb{R}^n)$.

viii) Show the converse, that each element of $\mathcal{C}^{-\infty}(\mathbb{R}^n)$ with compact support is an element of $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and hence conclude that $\mathcal{E}'(\mathbb{R}^n)$ 'is' the space of distributions of compact support.

I will denote the space of distributions of compact support by $\mathcal{C}_c^{-\infty}(\mathbb{R})$.

Problem 60. Hypoellipticity of the heat operator $H = iD_t + \Delta = iD_t + \sum_{j=1}^n D_{x_j}^2$ on $\mathbb{R}^{n+1}$.

(1) Using τ to denote the 'dual variable' to t and $\xi \in \mathbb{R}^n$ to denote the dual variables to $x \in \mathbb{R}^n$ observe that $H = p(D_t, D_x)$ where $p = i\tau + |\xi|^2$.

(2) Show that $|p(\tau, \xi)| > \frac{1}{2}\left(|\tau| + |\xi|^2\right)$.

(3) Use an inductive argument to show that, in $(\tau, \xi) \neq 0$ where it makes sense,

$$D_\tau^k D_\xi^\alpha \frac{1}{p(\tau,\xi)} = \sum_{j=1}^{|\alpha|} \frac{q_{k,\alpha,j}(\xi)}{p(\tau,\xi)^{k+j+1}} \tag{17.12}$$

where $q_{k,\alpha,j}(\xi)$ is a polynomial of degree (at most) $2j - |\alpha|$.

(4) Conclude that if $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ is identically equal to 1 in a neighbourhood of 0 then the function

$$g(\tau, \xi) = \frac{1 - \phi(\tau, \xi)}{i\tau + |\xi|^2}$$

is the Fourier transform of a distribution $F \in \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{sing\,supp}(F) \subset \{0\}$. [Remember that $\operatorname{sing\,supp}(F)$ is the complement of the largest open subset of $\mathbb{R}^n$ the restriction of F to which is smooth].

(5) Show that F is a parametrix for the heat operator.

(6) Deduce that $iD_t + \Delta$ is *hypoelliptic* – that is, if $U \subset \mathbb{R}^n$ is an open set and $u \in \mathcal{C}^{-\infty}(U)$ satisfies $(iD_t + \Delta)u \in \mathcal{C}^\infty(U)$ then $u \in \mathcal{C}^\infty(U)$.

(7) Show that $iD_t - \Delta$ is also hypoelliptic.

Problem 61. Wavefront set computations and more – all pretty easy, especially if you use results from class.

i) Compute $\mathrm{WF}(\delta)$ where $\delta \in \mathcal{S}'(\mathbb{R}^n)$ is the Dirac delta function at the origin.

ii) Compute $\mathrm{WF}(H(x))$ where $H(x) \in \mathcal{S}'(\mathbb{R})$ is the Heaviside function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

Hint: D_x is elliptic in one dimension, hit H with it.

iii) Compute $\mathrm{WF}(E)$, $E = iH(x_1)\delta(x')$ which is the Heaviside in the first variable on $\mathbb{R}^n$, $n > 1$, and delta in the others.

iv) Show that $D_{x_1}E = \delta$, so E is a fundamental solution of D_{x_1}.

v) If $f \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ show that $u = E \star f$ solves $D_{x_1}u = f$.

vi) What does our estimate on $\mathrm{WF}(E \star f)$ tell us about $\mathrm{WF}(u)$ in terms of $\mathrm{WF}(f)$?

Problem 62. The wave equation in two variables (or one spatial variable).

i) Recall that the Riemann function

$$E(t,x) = \begin{cases} -\frac{1}{4} & \text{if } t > x \text{ and } t > -x \\ 0 & \text{otherwise} \end{cases}$$

is a fundamental solution of $D_t^2 - D_x^2$ (check my constant).

ii) Find the singular support of E.

iii) Write the Fourier transform (dual) variables as τ, ξ and show that

$$\mathrm{WF}(E) \subset \{0\} \times \mathbb{S}^1 \cup \{(t,x,\tau,\xi); x = t > 0 \text{ and } \xi + \tau = 0\} \\ \cup \{(t,x,\tau,\xi); -x = t > 0 \text{ and } \xi = \tau\}.$$

iv) Show that if $f \in \mathcal{C}_c^{-\infty}(\mathbb{R}^2)$ then $u = E \star f$ satisfies $(D_t^2 - D_x^2)u = f$.

v) With u defined as in iv) show that

$$\mathrm{supp}(u) \subset \{(t,x);\ \exists \\ (t',x') \in \mathrm{supp}(f) \text{ with } t' + x' \le t + x \text{ and } t' - x' \le t - x\}.$$

vi) Sketch an illustrative example of v).

vii) Show that, still with u given by iv),

$$\operatorname{sing\,supp}(u) \subset \{(t,x);\ \exists\ (t',x') \in \operatorname{sing\,supp}(f) \text{ with} \\ t \ge t' \text{ and } t + x = t' + x' \text{ or } t - x = t' - x'\}.$$

viii) Bound $\mathrm{WF}(u)$ in terms of $\mathrm{WF}(f)$.

Problem 63. A little uniqueness theorems. Suppose $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ recall that the Fourier transform $\hat{u} \in \mathcal{C}^\infty(\mathbb{R}^n)$. Now, suppose $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ satisfies $P(D)u = 0$ for some non-trivial polynomial P, show that $u = 0$.

Problem 64. Work out the elementary behavior of the heat equation.

i) Show that the function on $\mathbb{R} \times \mathbb{R}^n$, for $n \ge 1$,

$$F(t,x) = \begin{cases} t^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4t}\right) & t > 0 \\ 0 & t \le 0 \end{cases}$$

is measurable, bounded on the any set $\{|(t,x)| \ge R\}$ and is integrable on $\{|(t,x)| \le R\}$ for any $R > 0$.

ii) Conclude that F defines a tempered distibution on $\mathbb{R}^{n+1}$.

iii) Show that F is $\mathcal{C}^\infty$ outside the origin.

iv) Show that F satisfies the heat equation

$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2) F(t,x) = 0 \text{ in } (t,x) \ne 0.$$

v) Show that F satisfies

$$F(s^2t, sx) = s^{-n}F(t,x) \text{ in } \mathcal{S}'(\mathbb{R}^{n+1}) \tag{17.13}$$

where the left hand side is defined by duality "$F(s^2t, sx) = F_s$" where

$$F_s(\phi) = s^{-n-2}F(\phi_{1/s}),\ \phi_{1/s}(t,x) = \phi(\frac{t}{s^2}, \frac{x}{s}).$$

vi) Conclude that

$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2)F(t,x) = G(t,x)$$

where $G(t,x)$ satisfies

$$G(s^2t, sx) = s^{-n-2}G(t,x) \text{ in } \mathcal{S}'(\mathbb{R}^{n+1}) \tag{17.14}$$

in the same sense as above and has support at most $\{0\}$.

vii) Hence deduce that

$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2)F(t,x) = c\delta(t)\delta(x) \tag{17.15}$$

for some real constant c.

Hint: Check which distributions with support at $(0,0)$ satisfy (17.14).

viii) If $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ show that $u = F \star \psi$ satisfies

$$u \in \mathcal{C}^\infty(\mathbb{R}^{n+1}) \text{ and} \tag{17.16}$$
$$\sup_{x\in\mathbb{R}^n,\ t\in[-S,S]} (1+|x|)^N |D^\alpha u(t,x)| < \infty\ \forall\ S > 0, \alpha \in \mathbb{N}^{n+1},\ N.$$

ix) Supposing that u satisfies (17.16) and is a real-valued solution of

$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2)u(t,x) = 0$$

in $\mathbb{R}^{n+1}$, show that

$$v(t) = \int_{\mathbb{R}^n} u^2(t,x)$$

is a non-increasing function of t.

Hint: Multiply the equation by u and integrate over a slab $[t_1, t_2] \times \mathbb{R}^n$.

x) Show that c in (17.15) is non-zero by arriving at a contradiction from the assumption that it is zero. Namely, show that if $c = 0$ then u in viii) satisfies the conditions of ix) and also vanishes in $t < T$ for some T (depending on ψ). Conclude that $u = 0$ for all ψ. Using properties of convolution show that this in turn implies that $F = 0$ which is a contradiction.

xi) So, finally, we know that $E = \frac{1}{c}F$ is a fundamental solution of the heat operator which vanishes in $t < 0$. Explain why this allows us to show that for any $\psi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}^n)$ there is a solution of

$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2)u = \psi,\ \ u = 0 \text{ in } t < T \text{ for some } T. \tag{17.17}$$

What is the largest value of T for which this holds?

xii) Can you give a heuristic, or indeed a rigorous, explanation of why

$$c = \int_{\mathbb{R}^n} \exp(-\frac{|x|^2}{4})dx?$$

xiii) Explain why the argument we used for the wave equation to show that there is *only one* solution, $u \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$, of (17.17) does not apply here. (Indeed such uniqueness does not hold without some growth assumption on u.)

Problem 65. (Poisson summation formula) As in class, let $L \subset \mathbb{R}^n$ be an integral lattice of the form

$$L = \left\{ v = \sum_{j=1}^n k_j v_j,\ k_j \in \mathbb{Z} \right\}$$

where the v_j form a basis of $\mathbb{R}^n$ and using the dual basis w_j (so $w_j \cdot v_i = \delta_{ij}$ is 0 or 1 as $i \neq j$ or $i = j$) set

$$L^\circ = \left\{ w = 2\pi \sum_{j=1}^n k_j w_j,\ k_j \in \mathbb{Z} \right\}.$$

Recall that we defined

$$\mathcal{C}^\infty(\mathbb{T}_L) = \{u \in \mathcal{C}^\infty(\mathbb{R}^n); u(z+v) = u(z)\ \forall\ z \in \mathbb{R}^n,\ v \in L\}. \tag{17.18}$$

i) Show that summation over shifts by lattice points:

$$A_L : \mathcal{S}(\mathbb{R}^n) \ni f \longmapsto A_L f(z) = \sum_{v \in L} f(z - v) \in \mathcal{C}^\infty(\mathbb{T}_L). \tag{17.19}$$

defines a map into smooth periodic functions.

ii) Show that there exists $f \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ such that $A_L f \equiv 1$ is the costant function on $\mathbb{R}^n$.
iii) Show that the map (17.19) is surjective. Hint: Well obviously enough use the f in part ii) and show that if u is periodic then $A_L(uf) = u$.
iv) Show that the infinite sum

$$F = \sum_{v \in L} \delta(\cdot - v) \in \mathcal{S}'(\mathbb{R}^n) \tag{17.20}$$

does indeed define a tempered distribution and that F is L-periodic and satisfies $\exp(iw \cdot z)F(z) = F(z)$ for each $w \in L^\circ$ with equality in $\mathcal{S}'(\mathbb{R}^n)$.
v) Deduce that $\hat{F}$, the Fourier transform of F, is L° periodic, conclude that it is of the form

$$\hat{F}(\xi) = c \sum_{w \in L^\circ} \delta(\xi - w) \tag{17.21}$$

vi) Compute the constant c.
vii) Show that $A_L(f) = F \star f$.
viii) Using this, or otherwise, show that $A_L(f) = 0$ in $\mathcal{C}^\infty(\mathbb{T}_L)$ if and only if $\hat{f} = 0$ on L°.

Problem 66. For a measurable set $\Omega \subset \mathbb{R}^n$, with non-zero measure, set $H = L^2(\Omega)$ and let $\mathcal{B} = \mathcal{B}(H)$ be the algebra of bounded linear operators on the Hilbert space H with the norm on $\mathcal{B}$ being

$$\|B\|_{\mathcal{B}} = \sup\{\|Bf\|_H ; f \in H,\ \|f\|_H = 1\}. \tag{17.22}$$

i) Show that $\mathcal{B}$ is complete with respect to this norm. Hint (probably not necessary!) For a Cauchy sequence $\{B_n\}$ observe that $B_n f$ is Cauchy for each $f \in H$.
ii) If $V \subset H$ is a finite-dimensional subspace and $W \subset H$ is a closed subspace with a finite-dimensional complement (that is $W + U = H$ for some finite-dimensional subspace U) show that there is a closed subspace $Y \subset W$ with finite-dimensional complement (in H) such that $V \perp Y$, that is $\langle v, y\rangle = 0$ for all $v \in V$ and $y \in Y$.
iii) If $A \in \mathcal{B}$ has finite rank (meaning AH is a finite-dimensional vector space) show that there is a finite-dimensional space $V \subset H$ such that $AV \subset V$ and $AV^\perp = \{0\}$ where

$$V^\perp = \{f \in H; \langle f, v\rangle = 0\ \forall\ v \in V\}.$$

Hint: Set $R = AH$, a finite dimensional subspace by hypothesis. Let N be the null space of A, show that $N^\perp$ is finite dimensional. Try $V = R + N^\perp$.

iv) If $A \in \mathcal{B}$ has finite rank, show that $(\operatorname{Id} - zA)^{-1}$ exists for all but a finite set of $\lambda \in \mathbb{C}$ (just quote some matrix theory). What might it mean to say in this case that $(\operatorname{Id} - zA)^{-1}$ is meromorphic in z? (No marks for this second part).

v) Recall that $\mathcal{K} \subset \mathcal{B}$ is the algebra of compact operators, defined as the closure of the space of finite rank operators. Show that $\mathcal{K}$ is an ideal in $\mathcal{B}$.

vi) If $A \in \mathcal{K}$ show that

$$\operatorname{Id} + A = (\operatorname{Id} + B)(\operatorname{Id} + A')$$

where $B \in \mathcal{K}$, $(\operatorname{Id} + B)^{-1}$ exists and A' has finite rank. Hint: Use the invertibility of $\operatorname{Id} + B$ when $\|B\|_{\mathcal{B}} < 1$ proved in class.

vii) Conclude that if $A \in \mathcal{K}$ then

$$\{f \in H; (\operatorname{Id} + A)f = 0\} \text{ and } \big((\operatorname{Id} + A)H\big)^\perp \text{ are finite dimensional.}$$

Problem 67. [Separable Hilbert spaces]

i) (Gramm-Schmidt Lemma). Let $\{v_i\}_{i\in\mathbb{N}}$ be a sequence in a Hilbert space H. Let $V_j \subset H$ be the span of the first j elements and set $N_j = \dim V_j$. Show that there is an orthonormal sequence $e_1, \dots, e_j$ (finite if N_j is bounded above) such that V_j is the span of the first N_j elements. Hint: Proceed by induction over N such that the result is true for all j with $N_j < N$. So, consider what happens for a value of j with $N_j = N_{j-1} + 1$ and add element $e_{N_j} \in V_j$ which is orthogonal to all the previous e_k's.

ii) A Hilbert space is separable if it has a countable dense subset (sometimes people say Hilbert space when they mean separable Hilbert space). Show that every separable Hilbert space has a complete orthonormal sequence, that is a sequence $\{e_j\}$ such that $\langle u, e_j\rangle = 0$ for all j implies $u = 0$.

iii) Let $\{e_j\}$ an orthonormal sequence in a Hilbert space, show that for any $a_j \in \mathbb{C}$,

$$\|\sum_{j=1}^N a_j e_j\|^2 = \sum_{j=1}^N |a_j|^2.$$

iv) (Bessel's inequality) Show that if e_j is an orthormal sequence in a Hilbert space and $u \in H$ then

$$\|\sum_{j=1}^{N}\langle u, e_j\rangle e_j\|^2 \le \|u\|^2$$

and conclude (assuming the sequence of e_j's to be infinite) that the series

$$\sum_{j=1}^{\infty}\langle u, e_j\rangle e_j$$

converges in H.

v) Show that if e_j is a complete orthonormal basis in a separable Hilbert space then, for each $u \in H$,

$$u = \sum_{j=1}^{\infty}\langle u, e_j\rangle e_j.$$

Problem 68. [Compactness] Let's agree that a compact set in a metric space is one for which every open cover has a finite subcover. You may use the compactness of closed bounded sets in a finite dimensional vector space.

i) Show that a compact subset of a Hilbert space is closed and bounded.

ii) If e_j is a complete orthonormal subspace of a separable Hilbert space and K is compact show that given $\epsilon > 0$ there exists N such that

$$\sum_{j\ge N}|\langle u, e_j\rangle|^2 \le \epsilon \;\forall\; u \in K. \tag{17.23}$$

iii) Conversely show that any closed bounded set in a separable Hilbert space for which (17.23) holds for some orthonormal basis is indeed compact.

iv) Show directly that any sequence in a compact set in a Hilbert space has a convergent subsequence.

v) Show that a subspace of H which has a precompact unit ball must be finite dimensional.

vi) Use the existence of a complete orthonormal basis to show that any bounded sequence $\{u_j\}$, $\|u_j\| \le C$, has a weakly convergent subsequence, meaning that $\langle v, u_j\rangle$ converges in $\mathbb{C}$ along the subsequence for each $v \in H$. Show that the subsequnce can be chosen so that $\langle e_k, u_j\rangle$ converges for each k, where e_k is the complete orthonormal sequence.

Problem 69. [Spectral theorem, compact case] Recall that a bounded operator A on a Hilbert space H is compact if $A\{\|u\| \leq 1\}$ is precompact (has compact closure). Throughout this problem A will be a compact operator on a separable Hilbert space, H.

i) Show that if $0 \neq \lambda \in \mathbb{C}$ then

$$E_\lambda = \{u \in H; Au = \lambda u\}.$$

is finite dimensional.

ii) If A is self-adjoint show that all eigenvalues (meaning $E_\lambda \neq \{0\}$) are real and that different eigenspaces are orthogonal.

iii) Show that $\alpha_A = \sup\{|\langle Au, u\rangle|^2\}; \|u\| = 1\}$ is attained. Hint: Choose a sequence such that $|\langle Au_j, u_j\rangle|^2$ tends to the supremum, pass to a weakly convergent sequence as discussed above and then using the compactness to a furhter subsequence such that Au_j converges.

iv) If v is such a maximum point and $f \perp v$ show that $\langle Av, f\rangle + \langle Af, v\rangle = 0$.

v) If A is also self-adjoint and u is a maximum point as in iii) deduce that $Au = \lambda u$ for some $\lambda \in \mathbb{R}$ and that $\lambda = \pm\alpha$.

vi) Still assuming A to be self-adjoint, deduce that there is a finite-dimensional subspace $M \subset H$, the sum of eigenspaces with eigenvalues $\pm\alpha$, containing all the maximum points.

vii) Continuing vi) show that A restricts to a self-adjoint bounded operator on the Hilbert space $M^\perp$ and that the supremum in iii) for this new operator is smaller.

viii) Deduce that for any compact self-adjoint operator on a separable Hilbert space there is a complete orthonormal basis of eigenvectors. Hint: Be careful about the null space – it could be big.

Problem 70. Show that a (complex-valued) square-integrable function $u \in L^2(\mathbb{R}^n)$ is continuous in the mean, in the sense that

$$\lim_{\epsilon \downarrow 0} \sup_{|y|<\epsilon} \int |u(x+y) - u(x)|^2 dx = 0. \tag{17.24}$$

Hint: Show that it is enough to prove this for non-negative functions and then that it suffices to prove it for non-negative simple functions and finally that it is enough to check it for the characteristic function of an open set of finite measure. Then use Problem 57 to show that it is true in this case.

Problem 71. [Ascoli-Arzela] Recall the proof of the theorem of Ascoli and Arzela, that a subset of $\mathcal{C}_0^0(\mathbb{R}^n)$ is precompact (with respect to the supremum norm) if and only if it is equicontinuous and equi-small at infinity, i.e. given $\epsilon > 0$ there exists $\delta > 0$ such that for all elements $u \in B$

(17.25)
$$|y| < \delta \Longrightarrow \sup_{x \in \mathbb{R}^n} |u(x+y) = u(x)| < \epsilon \text{ and } |x| > 1/\delta \Longrightarrow |u(x)| < \epsilon.$$

Problem 72. [Compactness of sets in $L^2(\mathbb{R}^n)$.] Show that a subset $B \subset L^2(\mathbb{R}^n)$ is *precompact* in $L^2(\mathbb{R}^n)$ if and only if it satisfies the following two conditions:

i) (Equi-continuity in the mean) For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathbb{R}^n} |u(x+y) - u(x)|^2 dx < \epsilon \ \forall \ |y| < \delta, \ u \in B. \tag{17.26}$$

ii) (Equi-smallness at infinity) For each $\epsilon > 0$ there exists R such that

$$\int_{|x|>R|} |u|^2 dx < \epsilon \ \forall \ u \in B. \tag{17.27}$$

Hint: Problem 70 shows that (17.26) holds for each $u \in L^2(\mathbb{R}^n)$; check that (17.27) also holds for each function. Then use a covering argument to prove that both these conditions must hold for a compact subset of $L^2(\mathbb{R})$ and hence for a precompact set. One method to prove the converse is to show that if (17.26) and (17.27) hold then B is bounded and to use this to extract a weakly convergent sequence from any given sequence in B. Next show that (17.26) is equivalent to (17.27) for the set $\mathcal{F}(B)$, the image of B under the Fourier transform. Show, possibly using Problem 71, that if χ_R is cut-off to a ball of radius R then $\chi_R \mathcal{G}(\chi_R \hat{u}_n)$ converges strongly if u_n converges weakly. Deduce from this that the weakly convergent subsequence in fact converges strongly so $\bar{B}$ is sequently compact, and hence is compact.

Problem 73. Consider the space $\mathcal{C}_c(\mathbb{R}^n)$ of all continuous functions on $\mathbb{R}^n$ with compact support. Thus each element vanishes in $|x| > R$ for some R, depending on the function. We want to give this a toplogy in terms of which is complete. We will use the *inductive limit* topology. Thus the whole space can be written as a countable union

(17.28)
$$\mathcal{C}_c(\mathbb{R}^n) = \bigcup_n \{u : \mathbb{R}^n; u \text{ is continuous and } u(x) = 0 \text{ for } |x| > R\}.$$

Each of the space on the right is a Banach space for the supremum norm.

(1) Show that the supreumum norm is not complete on the whole of this space.
(2) Define a subset $U \subset \mathcal{C}_{\mathrm{c}}(\mathbb{R}^n)$ to be open if its intersection with each of the subspaces on the right in (17.28) is open w.r.t. the supremum norm.
(3) Show that this definition does yield a topology.
(4) Show that any sequence $\{f_n\}$ which is 'Cauchy' in the sense that for any open neighbourhood U of 0 there exists N such that $f_n - f_m \in U$ for all $n, m \geq N$, is convergent (in the corresponding sense that there exists f in the space such that $f - f_n \in U$ eventually).
(5) If you are determined, discuss the corresponding issue for nets.

Problem 74. Show that the continuity of a linear functional $u : \mathcal{C}_c^\infty(\mathbb{R}^n) \longrightarrow \mathbb{C}$ with respect to the inductive limit topology defined in (6.16) means precisely that for each $n \in \mathbb{N}$ there exists $k = k(n)$ and $C = C_n$ such that

$$|u(\varphi)| \leq C\|\varphi\|_{\mathcal{C}^k}, \ \forall \ \varphi \in \dot{\mathcal{C}}^\infty(B(n)). \tag{17.29}$$

The point of course is that the 'order' k and the constnat C can both increase as n, measuring the size of the support, increases.

Problem 75. [Restriction from Sobolev spaces] The Sobolev embedding theorem shows that a function in $H^m(\mathbb{R}^n)$, for $m > n/2$ is continuous – and hence can be restricted to a subspace of $\mathbb{R}^n$. In fact this works more generally. Show that there is a well defined *restriction map*

$$H^m(\mathbb{R}^n) \longrightarrow H^{m-\frac{1}{2}}(\mathbb{R}^n) \text{ if } m > \frac{1}{2} \tag{17.30}$$

with the following properties:

(1) On $\mathcal{S}(\mathbb{R}^n)$ it is given by $u \longmapsto u(0, x')$, $x' \in \mathbb{R}^{n-1}$.
(2) It is continuous and linear.

Hint: Use the usual method of finding a weak version of the map on smooth Schwartz functions; namely show that in terms of the Fourier transforms on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$

$$\widehat{u(0,\cdot)}(\xi') = (2\pi)^{-1} \int_{\mathbb{R}} \hat{u}(\xi_1, \xi') d\xi_1, \ \forall \ \xi' \in \mathbb{R}^{n-1}. \tag{17.31}$$

Use Cauchy's inequality to show that this is continuous as a map on Sobolev spaces as indicated and then the density of $\mathcal{S}(\mathbb{R}^n)$ in $H^m(\mathbb{R}^n)$ to conclude that the map is well-defined and unique.

Problem 76. [Restriction by WF] From class we know that the product of two distributions, one with compact support, is defined provided they have no ‘opposite’ directions in their wavefront set:

$$(x,\omega)\in \mathrm{WF}(u) \Longrightarrow (x,-\omega)\notin \mathrm{WF}(v) \text{ then } uv\in \mathcal{C}_c^{-\infty}(\mathbb{R}^n). \tag{17.32}$$

Show that this product has the property that $f(uv)=(fu)v=u(fv)$ if $f\in \mathcal{C}^\infty(\mathbb{R}^n)$. Use this to define a restriction map to $x_1=0$ for distributions of compact support satisfying $((0,x'),(\omega_1,0))\notin \mathrm{WF}(u)$ as the product

$$u_0 = u\delta(x_1). \tag{17.33}$$

[Show that $u_0(f)$, $f\in\mathcal{C}^\infty(\mathbb{R}^n)$ only depends on $f(0,\cdot)\in \mathcal{C}^\infty(\mathbb{R}^{n-1})$.

Problem 77. [Stone’s theorem] For a bounded self-adjoint operator A show that the spectral measure can be obtained from the resolvent in the sense that for $\phi,\psi\in H$

$$\lim_{\epsilon\downarrow 0}\frac{1}{2\pi i}\langle[(A-t-i\epsilon)^{-1}-(A+t+i\epsilon)^{-1}]\phi,\psi\rangle \longrightarrow \mu_{\phi,\psi} \tag{17.34}$$

in the sense of distributions – or measures if you are prepared to work harder!

Problem 78. If $u\in\mathcal{S}(\mathbb{R}^n)$ and $\psi'=\psi_R+\mu$ is, as in the proof of Lemma 12.5, such that

$$\mathrm{supp}(\psi')\cap \mathrm{Css}(u)=\emptyset$$

show that

$$\mathcal{S}(\mathbb{R}^n)\ni\phi\longmapsto \phi\psi' u\in\mathcal{S}(\mathbb{R}^n)$$

is continuous and hence (or otherwise) show that the functional u_1u_2 defined by (12.20) is an element of $\mathcal{S}'(\mathbb{R}^n)$.

Problem 79. Under the conditions of Lemma 12.10 show that

$$\mathrm{Css}(u*v)\cap\mathbb{S}^{n-1}\subset\{\frac{sx+ty}{|sx+ty|},\ |x|=|y|=1,\ x\in\mathrm{Css}(u),\ y\in\mathrm{Css}(v),\ 0\le s,t\le 1\}. \tag{17.35}$$

Notice that this make sense exactly because $sx+ty=0$ implies that $t/s=1$ but $x+y\neq 0$ under these conditions by the assumption of Lemma 12.10.

Problem 80. Show that the pairing $u(v)$ of two distributions $u,v\in {}^{\mathrm{b}}S'(\mathbb{R}^n)$ may be defined under the hypothesis (12.50).

Problem 81. Show that under the hypothesis (12.51)

(17.36)
$$\begin{aligned}\mathrm{WF}_{\mathrm{sc}}(u*v) \subset \{(x+y,p);(x,p)\in \mathrm{WF}_{\mathrm{sc}}(u)\cap(\mathbb{R}^n\times\mathbb{S}^{n-1}),\ (y,p)\in\mathrm{WF}_{\mathrm{sc}}(v)\cap(\mathbb{R}^n\times\mathbb{S}^{n-1})\}\\ \cup\{(\theta,q)\in\mathbb{S}^{n-1}\times\mathbb{B}^n;\theta=\frac{s'\theta'+s''\theta''}{|s'\theta'+s''\theta''|},\ 0\le s',s''\le 1,\\ (\theta',q)\in\mathrm{WF}_{\mathrm{sc}}(u)\cap(\mathbb{S}^{n-1}\times\mathbb{B}^n),\ (\theta'',q)\in\mathrm{WF}_{\mathrm{sc}}(v)\cap(\mathbb{S}^{n-1}\times\mathbb{B}^n)\}.\end{aligned}$$

Problem 82. Formulate and prove a bound similar to (17.36) for $\mathrm{WF}_{\mathrm{sc}}(uv)$ when u, $v\in\mathcal{S}'(\mathbb{R}^n)$ satisfy (12.50).

Problem 83. Show that for convolution $u*v$ defined under condition (12.51) it is still true that

$$P(D)(u*v)=(P(D)u)*v=u*(P(D)v). \tag{17.37}$$

Problem 84. Using Problem 80 (or otherwise) show that integration is defined as a functional

$$\{u\in\mathcal{S}'(\mathbb{R}^n);(\mathbb{S}^{n-1}\times\{0\})\cap\mathrm{WF}_{\mathrm{sc}}(u)=\emptyset\}\longrightarrow\mathbb{C}. \tag{17.38}$$

If u satisfies this condition, show that $\int P(D)u=c\int u$ where c is the constant term in $P(D)$, i.e. $P(D)1=c$.

Problem 85. Compute $\mathrm{WF}_{\mathrm{sc}}(E)$ where $E=C/|x-y|$ is the standard fundamental solution for the Laplacian on $\mathbb{R}^3$. Using Problem 83 give a condition on $\mathrm{WF}_{\mathrm{sc}}(f)$ under which $u=E*f$ is defined and satisfies $\Delta u=f$. Show that under this condition $\int f$ is defined using Problem 84. What can you say about $\mathrm{WF}_{\mathrm{sc}}(u)$? Why is it not the case that $\int\Delta u=0$, even though this is true if u has compact support?