Lecture 15

April 8th, 2004

The Continuity Method

$\mathcal{L}$et $T : \mathcal{B}_1 \to \mathcal{B}_2$ be linear between two Banach spaces. T is *bounded* if

$$||T|| = \sup_{x\in\mathcal{B}_1} \frac{||Tx||_{\mathcal{B}_2}}{||x||_{\mathcal{B}_1}} < \infty \quad \Leftrightarrow \quad ||Tx||_{\mathcal{B}_2} \leq c \cdot ||x||_{\mathcal{B}_1} \text{ for some } c > 0.$$

Continuity Method Theorem. *Let $\mathcal{B}$ be a Banach space , V a normed space, $L_0, L_1 : \mathcal{B} \to V$ bounded linear operators. Assume $\exists c$ such that $L_t := (1-t)L_0 + tL_1$ satisfies*

$$||x||_{\mathcal{B}} \leq c \cdot ||L_t x||_V, \quad \forall t \in [0,1]. \tag{$*$}$$

Then – L_0 is onto $\Leftrightarrow$ L_1 is.

Proof. Assume L_s is onto for some $s \in [0,1]$; by $(*)$ L_s is also 1-to-1 $\Rightarrow L_s^{-1}$ exists. For $t \in [0,1], y \in V$ solving $L_t x = y$ is equivalent to solving $L_s(x) = y + (L_s - L_t)x = y + (t-s)L_0 x + (t-s)L_1 x$. By linearity now $x = L_s^{-1}y + (t-s){L_s}^{-1} \circ (L_0 - L_1)x$.

Define a linear map $T : \mathcal{B} \to \mathcal{B}$, $Tx = L_s^{-1}y + (t-s){L_s}^{-1} \circ (L_0 - L_1)x$. One has $||Tx_1 - Tx_2||_{\mathcal{B}} = ||(t-s){L_s}^{-1} \circ (L_0 - L_1)(x_1 - x_2)||$. $(*)$ now gives us a bound on ${L_s}^{-1}$: since L_s is onto $\forall x \in \mathcal{B}$, $\exists y \in \mathcal{B}$ such that $L_s y = x$ and so

$$||{L_s}^{-1}x||_{\mathcal{B}} \leq c \cdot ||L_s \circ {L_s}^{-1}x||_V$$

$$||{L_s}^{-1}x||_{\mathcal{B}} \leq c \cdot ||x||_V \quad \Rightarrow \quad ||{L_s}^{-1}|| \leq c.$$

As an application we see that

$$||Tx_1 - Tx_2||_{\mathcal{B}} \leq (t-s)c \cdot (||L_0|| + ||L_1||)||x_1 - x_2||,$$

and for t close enough to s (precisely for $t \in [s - \frac{1}{c(||L_0||+||L_1||)}, s + \frac{1}{c(||L_0||+||L_1||)}]$) we therefore have a contraction mapping! Therefore T has a fixed point by the previous theorem which essentially means that we can solve $L_t x = y$ for any fixed y or that L_t is onto. Repeating this $c(||L_0|| + ||L_1||)$ many times we cover all $t \in [0, 1]$. ■

Remark. Note as in the beginning of the proof that once such operators are onto they are in fact invertible as long as $(*)$ holds.

Elliptic uniqueness

Let us summarize the properties we have establised for uniformly elliptic equations. Let Ω be a bounded domain in $\mathbb{R}^n$. Let $L = a^{ij}(x)\mathrm{D}_{ij} + b^i(x)\mathrm{D}_i + c(x)$ be uniformly elliptic, i.e

$$\frac{1}{\Lambda} \cdot \delta^{ij} \leq a^{ij}(x) \leq \Lambda \cdot \delta^{ij}$$

and assume $c(x) \leq 0$.

Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ be a solution of $Lu = f \in \mathcal{C}^\alpha(\Omega)$ with $0 < \alpha < 1$. Then we have the following a priori estimates –

A. $\sup_\Omega |u| \leq c(\gamma, \Lambda, \Omega, n) \cdot (\sup_{\partial\Omega} |u| + \sup_\Omega |f|)$.

B. Under the additional assumptions

- in the case L has $\alpha-$ Hölder continuous coefficients with Hölder constant Λ,
- Ω has $\mathcal{C}^{2,\alpha}$ boundary
- $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega}), f \in \mathcal{C}^\alpha(\bar{\Omega})$,

we had the global Schauder estimate

$$||u||_{C^{2,\alpha}(\bar{\Omega})} \leq c(\gamma, \Lambda, \Omega, n)\big(||u||_{C^0(\Omega)} + ||f||_{C^\alpha(\Omega)}\big).$$

C. Under the assumptions of B, when $c(x) \leq 0$

$$||u||_{C^{2,\alpha}(\bar{\Omega})} \leq c(\sup_{\partial\Omega} |u| + \sup_{\Omega} |f|).$$

D. The above applies to the Dirichlet problem

$$Lu = f \text{ on } \bar{\Omega}, \quad u = \varphi \text{ on } \partial\Omega$$

and in particular when $\varphi = 0$ we get very simply

$$||u||_{C^{2,\alpha}(\bar{\Omega})} \leq c \cdot ||Lu||_{C^{\alpha}(\bar{\Omega})}.$$

Theorem. *Let Ω be a $\mathcal{C}^{2,\alpha}$ domain, L uniformly elliptic with $\mathcal{C}^{\alpha}(\bar{\Omega})$ coefficients and $(x) \leq 0$. Look at all $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ and assume $f \in \mathcal{C}^{\alpha}(\bar{\Omega})$. Then the Dirichlet problem $Lu = f$ on $\bar{\Omega}$, $u = \varphi$ on $\partial\Omega$ has a unique solution $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ provided that the Dirichlet problem for Δ is solvable $\forall f \in \mathcal{C}^{\alpha}(\bar{\Omega})$, $\forall \varphi \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$!*

Proof. Connect L and Δ via a segment: $[0,1] \to L_t := (1-t)L + t\Delta$. Since those operators are all linear it is enough to prove for $\varphi = 0$ as we have seen previously. $\mathcal{C}^{2,\alpha}(\bar{\Omega})$ is a Banach space (Lecture 14), and so is its subspace $\mathcal{B}(\Omega) := \{u \in \mathcal{C}^{2,\alpha}(\bar{\Omega}), u = 0 \text{ on } \partial\Omega\}$. As a matter of fact L_t is a bounded operator $\mathcal{B}(\Omega) \to \mathcal{C}^{\alpha}(\bar{\Omega})$ by the assumptions on the coefficients of L. And, by uniformly elliptic we see from D above

$$||u||_{C^{2,\alpha}(\bar{\Omega})} = ||u||_{C^{2,\alpha}(\mathcal{B}(\Omega))} \leq c \cdot ||L_t u||_{C^{\alpha}(\bar{\Omega})},$$

with c independent of t (depends just on L). Note $\mathcal{C}^{\alpha}(\bar{\Omega})$ is a Banach space and in particular a vector space. The Continuity Method thus applies. ■

Strangely enough, we are now back to solving Dirichlet's problem for Δ in domains.

Our methods so far were good for providing solution in balls, spherically symmetric domains. In other words we were able to solve (in $\mathcal{C}^{2,\alpha}(\overline{B(0,R)})$!) $\Delta u = f \in \mathcal{C}^{\alpha}(\bar{\Omega})$ on $B(0,R)$, $u = \varphi$ on $\partial B(0,R)$ using the Poisson Integral Formula and estimates for the Newtonian Potential. We used conformal mappings (inversion) to get indeed $\mathcal{C}^{2,\alpha}$ upto the boundary. We conclude therefore that

Corollary. *We can solve the Dirichlet Problem for any L satisfying the assumptions of the Theorem in balls.*

Perron's Method gives a solution in quite general domains but we will not go into its details as later on our regularity theory (weak solutions, Sobolev spaces etc.) will give us those answers.

Elliptic $\mathcal{C}^{2,\alpha}$ regularity

Let B :=ball, T :=some connected boundary portion.

Theorem. *Let L be uniformly elliptic with $\mathcal{C}^\alpha$ coefficients and assume $c(x) \leq 0$. Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ be a solution of the Dirichlet problem $Lu = f \in \mathcal{C}^\alpha(\bar{B})$ in B, $\quad u = \varphi \in \mathcal{C}^0(\partial B) \cap \mathcal{C}^{2,\alpha}(T)$ on ∂B has a unique solution $u \in \mathcal{C}^{2,\alpha}(B \cup T) \cap \mathcal{C}^0(\bar{B})$.*

We know by the previous theorem that if $\varphi \in \mathcal{C}^{2,\alpha}(\partial B)$ (and not just on T) then unique solvability would be equivalent to the unique solvability of Δ on B which we have! Therefore this Theorem is a slight generalization.

Proof. As was just outlined the crucial problem lies in the (possible) absence of regularity of φ on part of the boundary. So we approximate φ by a sequence $\{\varphi_k\} \subset \mathcal{C}^3(\bar{B})$ such that both $||\varphi_k - \varphi||_{C^0(\bar{B})} \longrightarrow 0$ and $||\varphi_k - \varphi||_{C^{2,\alpha}(\bar{B})} \longrightarrow 0$. Solve $Lu_k = f$, in B, $\quad u_k = \varphi_k$on ∂B.

Now $L(u_i - u_j) = 0$, in B, $\quad u_i - u_j = \varphi_i - \varphi_j$ on ∂B. And by A above (as $c(x) \leq 0$) $||u_i - u_j||_{C^0(B)} \leq C \sup_{\partial B} |\varphi_i - \varphi_j|$. So we conclude our solutions $\{u_k\}$ form a Cauchy sequence WRT the $\mathcal{C}^0$ norm, i.e in the Banach space $\mathcal{C}^0(B)$. Therefore we know $\exists u \in \mathcal{C}^0(B)$ with $u_i \overset{\mathcal{C}^0(B)}{\longrightarrow} u$ (not just subconvergence!) and furthermore this u satisfies $u = \varphi$ on pB.

Now we shift our look to the $\mathcal{C}^{2,\alpha}$ situation; by our interior estimates we have for any $B' \Subset B$ $||u_i - u_j||_{C^{2,\alpha}(B')} \leq c(||u_i - u_j||_{C^0(B)} + ||0||_{C^\alpha(B)})$.. That is our sequence is also a Cauchy sequence in the Banach space $\mathcal{C}^{2,\alpha}(B') \;\Rightarrow\;$ converges in $\mathcal{C}^{2,\alpha}(B')$ (in particular limit is $\mathcal{C}^{2,\alpha}$ regular). This limit must equal the limit $u|'_B$ we obtained through the $\mathcal{C}^0$ norm. We do this for any $B' \Subset B \;\Rightarrow$ get convergence in $\mathcal{C}^{2,\alpha}(B) \;\Rightarrow\; u$ satisfies $Lu = f$ on B and has the desired $\mathcal{C}^{2,\alpha}$ regularity on B.

We now turn to the boundary portion: $\forall x_0 \in T$ and $\rho > 0$ such that $B(x_0, \rho) \cap \partial B \subseteq T$ we have the usual boundary Schauder estimates (for smooth enough functions) which give us $||u_i - u_j||_{C^{2,\alpha}(B(x_0,\rho)\cap\bar{B})} \leq c \cdot \left(||u_i - u_j||_{C^0(B)} + ||\varphi_i - \varphi_j||_{C^{2,\alpha}(B(x_0,\rho)\cap\bar{B})}\right)$. This means that in fact $u_i \xrightarrow{\mathcal{C}^{2,\alpha}(B(x_0,\rho)\cap\bar{B})} u$ and in particular $u \in \mathcal{C}^{2,\alpha}$ at x_0. $\forall x_0 \in T$. ■