Lecture 16

April 13th, 2004

Elliptic regularity

$\mathcal{H}$itherto we have always assumed our solutions already lie in the appropriate $\mathcal{C}^{k,\alpha}$ space and then showed estimates on their norms in those spaces. Now we will avoid this a priori assumption and show that they do hold a posteriori. This is important for the consistency of our discussion. Precisely what we would like to show is —

$\mathcal{A}$ priori regularity. *Let $u \in \mathcal{C}^2(\Omega)$ be a solution of $Lu = f$ and assume $0 < \alpha < 1$. We do not assume $c(x) \leq 0$ but we do assume all the other assumptions on L in the previous Theorem hold. If $f \in \mathcal{C}^\alpha(\Omega)$ then $u \in \mathcal{C}^{2,\alpha}(\Omega)$*

- Here we mean the $\mathcal{C}^\alpha$ norm is locally bounded, i.e for every point exists a neighborhood where the $\mathcal{C}^\alpha$-norm is bounded. Had we written $\mathcal{C}^\alpha(\bar{\Omega})$ we would mean a global bound on $\sup_{x,y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$ (as in the footnote if Lecture 14).

- This result will allow us to assume in previous theorems only $\mathcal{C}^2$ regularity on (candidate) solutions instead of assuming $\mathcal{C}^{2,\alpha}$ regularity.

Proof. Let u be a solution as above. Since the Theorem is *local* in nature we take any point in Ω and look at a ball B centered there contained in Ω. We then consider the Dirichlet problem

$$\begin{array}{rclcc} L_0 v & = & f' & \text{on} & B, \\ v & = & u & \text{on} & \partial B. \end{array}$$

where $L_0 := L - c(x)$ and $f'(x) := f(x) - c(x) \cdot u(x)$. This Dirichlet problem is on a ball, with "$c \leq 0$", uniform elliptic and with coefficients in $\mathcal{C}^\alpha$. Therefore we have uniqueness and existence of a solution v in $\mathcal{C}^{2,\alpha}(B) \cap \mathcal{C}^0(\bar{B})$. But u satisfies $Lu = f$ or equivalently $L_0 u = f'$ on all of Ω so in particular on $\bar{B}$. By uniqueness on B therefore we have $u|_{\bar{B}} = v$, and so u is $\mathcal{C}^{2,\alpha}$ smooth there. As this is for any point and all balls we have $u \in \mathcal{C}^{2,\alpha}(\Omega)$. ∎

It is insightful to note at this point that these results are optimal under the above assumptions. Indeed need $\mathcal{C}^2$ smoothness (or atleast $\mathcal{C}^{1,1}$) in order to define 2^{nd} derivatives WRT L! If one takes u in a larger function space, i.e weaker regularity of u, and defines $Lu = f$ in a weak sense then need more regularity on coefficients of L! Under the assumption of $\mathcal{C}^\alpha$ continuity on the coefficients indeed we are in an optimal situation.

$\mathcal{H}$igher a priori regularity. *Let $u \in \mathcal{C}^2(\Omega)$ be a solution of $Lu = f$ and $0 < \alpha < 1$. We do not assume $c(x) \leq 0$ but we assume uniformly elliptic and that all coefficients are in $\mathcal{C}^{k,\alpha}$. If $f \in \mathcal{C}^{k,\alpha}$ then $u \in \mathcal{C}^{k+2,\alpha}$. If $f \in \mathcal{C}^\infty$ then $u \in \mathcal{C}^\infty$.*

Proof. $k = 0$ was the previous Theorem.

The case $k = 1$. The proof relies in an elegant way on our previous results with the combination of the new idea of using *difference quotients*. We would like to differentiate the u three times and prove we get a $\mathcal{C}^\alpha$ function. Differentiating the equation $Lu = f$ once would serve our purpose but it can not be done naïvely as it would involve 3 derivatives of u and we only know that u has two. To circumvent this hurdle we will take two derivatives of the difference quotients of u, which we define by (let $\mathbf{e_1}, \ldots, \mathbf{e_n}$ denote the unit vectors in $\mathbb{R}^n$)

$$\Delta^h u := \frac{u(x + h \cdot \mathbf{e_l}) - u(x)}{h} =: .\frac{u^h(x) - u(x)}{h}.$$

Namely we look at

$$\Delta^h Lu = \frac{Lu(x + h \cdot \mathbf{e_l}) - Lu(x)}{h} = \frac{f(x + h \cdot \mathbf{e_l}) - f(x)}{h} = \Delta^h uf.$$

Note $\Delta^h v(x) \xrightarrow{h \to 0} \mathrm{D}_l v(x)$ *if* $v \in \mathcal{C}^1$ (which we don't know a priori in our case yet).

Expanding our equation in full gives

$$\frac{1}{h}\Big[(a^{ij}(x+h\cdot\mathbf{e_l})-a^{ij}(x)+a^{ij}(x))\mathrm{D}_{ij}u^h-a^{ij}(x)\mathrm{D}_{ij}u(x)$$

$$+b^i(x+h\cdot\mathbf{e_l})\mathrm{D}_i u(x+h\cdot\mathbf{e_l})-b^i(x)\mathrm{D}_i u(x)+c(x+h\cdot\mathbf{e_l})u(x+h\cdot\mathbf{e_l})-c(x)u(x)\Big]$$

$$=\Delta^h a^{ij}\mathrm{D}_{ij}u^h-a^{ij}\mathrm{D}_{ij}\Delta^h u+\Delta^h b^i\mathrm{D}_i u^h+b^i\mathrm{D}_i\Delta^h u+\Delta^h c\cdot u^h+c\cdot\Delta^h u=\Delta^h f.$$

or succintly

$$L\Delta^h u=f':=\Delta^h f-\Delta^h a^{ij}\cdot\mathrm{D}_{ij}u^h-\Delta^h b_i\cdot\mathrm{D}_i u^h-\Delta^h c\cdot u^h$$

where $u^h:=u(x+h\cdot\mathbf{e_1})$.

We now analyse the regularity of the terms. $f\in\mathcal{C}^{1,\alpha}$ so so is $\Delta^h f$, but not (bounded) uniformly WRT h (i.e $\mathcal{C}^{1,\alpha}$ norm of $\Delta^h f$ may go to ∞ as h decreases). On the otherhand $\Delta^h f\in\mathcal{C}^\alpha(\Omega)$ uniformly WRT h ($\forall h>0$): $\Delta^h uf=\frac{f(x+h\cdot\mathbf{e}_l)-f(x)}{h}=\mathrm{D}_l f(\bar{x})$ for some $\bar{x}$ in the interval, and RHS has a uniform $\mathcal{C}^\alpha$ bound as $f\in\mathcal{C}^{1,\alpha}$ on all Ω! (needed as $\bar{x}$ can be arbitrary).

For the same reason $\Delta^h a^{ij},\Delta^h b_i,\Delta^h c\in\mathcal{C}^\alpha(\Omega)$. By the $k=0$ case we know $u\in\mathcal{C}^{2,\alpha}(\Omega)$ and not just in $\mathcal{C}^2(\Omega)$. $\Leftrightarrow$ $\mathrm{D}_{ij}u^h\in\mathcal{C}^\alpha(\Omega)$ uniformly.

Remark. We take a moment to describe what we mean by uniformity. We say a function $g_h=g(h,\cdot):\Omega\to\mathbb{R}$ is *uniformly bounded in* $\mathcal{C}^\alpha$ WRT h when $\forall\Omega'\Subset\Omega$ exists $c(\Omega)$ such that $|g_h|_{C^\alpha(\Omega')}\le c(\Omega)$. Note this definition goes along with our *local* definition of a function being in $\mathcal{C}^\alpha(\Omega)$ (and not in $\mathcal{C}^\alpha(\bar{\Omega})$!).

Putting the above facts together we now see that both sides of the equation $L\Delta^h u=f'$ are in $\mathcal{C}^\alpha(\Omega)$. And they are also in $\mathcal{C}^\alpha(\Omega')$ with RHS uniformly so with constant $c(\Omega')$.

By the interior Schauder estimate, $\forall\Omega''\Subset\Omega'$ and for each h

$$||\Delta^h u||_{C^{2,\alpha}(\Omega'')}\le c(\gamma,\Lambda,\Omega'')\cdot\big(||\Delta^h u||_{C^{0,\Omega'}(+)}||f'||_{C^{\alpha,\Omega'}()}\big)\le\tilde{c}(\gamma,\Lambda,\Omega'',\Omega',\Omega,||u||_{C^1(\Omega)},$$

which is independent of h! If we assume the Claim below taking the limit $h\to 0$ we get $\mathrm{D}_l u\in\mathcal{C}^{2,\alpha}(\Omega''),\forall l=1,\dots,n$ $u\in\mathcal{C}^{3,\alpha}(\Omega'')$. $\forall\Omega''\Subset\Omega'\Subset\Omega\quad\Leftrightarrow\quad u\in\mathcal{C}^{3,\alpha}(\Omega)$. ∎

Claim. $||\Delta^h g||_{C^\alpha(A)} \leq c$ *independently of* $h \quad \Leftrightarrow \quad D_l g \in \mathcal{C}^\alpha(A)$.

First we we show $g \in \mathcal{C}^{0,1}(A)$. This is tantamount to the existence of $\lim_{h\to 0} \Delta^h g(x)$ (since if it exists it equals $\mathrm{D}_l u\gamma(x)$ - that's how we define the first l-directional derivative at x). Now $\{\Delta^h g\}_{h>0}$ is family of uniformly bounded (in $\mathcal{C}^0(A)$) and equicontinuous functions (from the uniform Hölder constant). So by the Arzelà-Ascoli Theorem exists a sequence $\{\Delta^{h_i} g\}_{i=1}^\infty$ converging to some $\tilde{w} \in \mathcal{C}^\alpha(A)$ in the $\mathcal{C}^\beta(A)$ norm for any $\beta < \alpha$. But as we remarked above $\tilde{w}$ necessarily equals $\mathrm{D}_l g$ by definition.

Second, we show $g \in \mathcal{C}^1(A)$ (i.e such that derivative is continuous not just bounded) and actually $\in \mathcal{C}^{1,\alpha}(A)$:

$$c \geq ||\Delta^h g||_{C^\alpha(A)} \geq \lim_{h\to 0} \frac{\Delta^h g(x) - \Delta^h g(y)}{|x-y|^\alpha} = \frac{\mathrm{D}_l g(x) - \mathrm{D}_l g(y)}{|x-y|^\alpha} = |\mathrm{D}_l g|_{C^\alpha(A)}$$

where we used that c is independent of h. ■

The case $k \geq 2$. Let $k = 2$. By the $k = 1$ case we can legitimately take 3 derivatives as $u \in \mathcal{C}^{3,\alpha}(\Omega)$. One has

$$L(\mathrm{D}_l u) = f' := \mathrm{D}_l f - \mathrm{D}_l a^{ij} \cdot \mathrm{D}_{ij} u - \mathrm{D}_l b_i \cdot \mathrm{D}_i u - \mathrm{D}_l c \cdot u$$

with $\mathrm{D}_l u, f' \in \mathcal{C}^{1,\alpha}(\Omega)$. So again by the $k = 1$ case we have now $\mathrm{D}_l u \in \mathcal{C}^{3,\alpha}(\Omega)$, hence $u \in \mathcal{C}^{4,\alpha}(\Omega)$. The instances $k \geq 3$ are in the same spirit. ■

Boundary regularity

Let Ω be a $\mathcal{C}^{2,\alpha}$ domain, i.e whose boundary is locally the graph of a $\mathcal{C}^{2,\alpha}$ function. Let L be uniformly elliptic with $\mathcal{C}^\alpha$ coefficients and $c \leq 0$.

Theorem. *Let* $f \in \mathcal{C}^\alpha(\Omega), \varphi \in \mathcal{C}^{2,\alpha}(\partial\Omega), u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ *satisfying* $\begin{array}{cccc} Lu & = & f & \text{on} & \Omega, \\ u & = & \varphi & \text{on} & \partial\partial\Omega. \end{array}$ *with* $0 < \alpha < 1$. *Then* $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$.

Proof. Our previous results give $u \in \mathcal{C}^{2,\alpha}(\Omega)$ and we seek to extend it to those points in $\partial\Omega$. Note that even though $u = \varphi$ on $\partial\Omega$ and φ is $\mathcal{C}^{2,\alpha}$ there this does not give the same property for u. It just gives that u is $\mathcal{C}^{2,\alpha}$ in directions tangent to $\partial\Omega$, but not in directions leading to the boundary.

The question is local: restrict attention to $B(x_0, R) \cap \bar{\Omega}$ for each $x_0 \in \partial\Omega$. We choose a $\mathcal{C}^{2,\alpha}$ homeomorphism $\Psi_1 : \mathbb{R}^n \to \mathbb{R}^n$ sending $B(x_0, R) \cap \partial\Omega$ to a portion of a (flat) hyperplane and $\partial B(x_0, R) \cap \Omega$ to the boundary of half a disc. We then choose another $\mathcal{C}^{2,\alpha}$ homeomorphism $\Psi_2 : \mathbb{R}^n \to \mathbb{R}^n$ sending the whole half disc into a disc (= a ball). Therefore $\Psi_2 \circ \Psi_1$ maps our original boundary portion into a portion of the boundary of a ball.

Similarly to previous computations of this sort we define the induced operator $\tilde{L}$ on the induced domain $\Psi_2 \circ \Psi_1(B(x_0, R) \cap \Omega)$ and define the induced functions $\tilde{u}, \tilde{\varphi}, \tilde{f}$ and we get a new Dirichlet problem with all norms of our original objects equivalent to those of our induced ones. Note that still $\tilde{c} := c \circ {\Psi_1}^{-1} \circ {\Psi_2}^{-1} \leq 0$, therefore by our theory exists a unique solution $v \in \mathcal{C}^{2,\alpha}\big(\Psi_2 \circ \Psi_1(B(x_0, R) \cap \Omega) \cup \Psi_2 \circ \Psi_1(B(x_0, R) \cap \partial\Omega)\big) \cap \mathcal{C}^0\big(\Psi_2 \circ \Psi_1(\overline{B(x_0, R)} \cap \bar{\Omega})\big)$ for the induced Dirichlet problem . Now our $\tilde{u}$ also solves it. So by uniqueness $\tilde{u} = v$ and $\tilde{u}$ has $\mathcal{C}^{2,\alpha}$ regularity as the induced boundary portion, and by pulling back through $\mathcal{C}^{2,\alpha}$ diffeomorphisms we get that so does u. ■

Remark. The assumption $c \leq 0$ is not necessary although modifying the proof is non-trivial without this assumption (exercise). We needed it in order to be able to use our existence result. But since we already assume a solution exists we may use some of our previous results which do not need $c \leq 0$ and which secure $\mathcal{C}^{2,\alpha}$ regularity upto the boundary.